Dynamical behavior of 2D positive systems

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Abstract

In the paper the definition and main properties of a 2-digraph, i.e. a directed graph with two kinds of arcs, are introduced. Natural constrains on the composition of the paths connecting each pair of vertices lead to the definition of 2-strongly connected digraph and of 2-imprimitivity classes.
Irreducible matrix pairs, that is pairs endowed with a 2-strongly connected digraph, are subsequently discussed. Equivalent descriptions of irreducibility, referring to the free evolution of the 2D state models described by the pairs and to their characteristic polynomials, are provided. Finally, primitivity, viewed as a special case of irreducibility, is introduced and characterized.

1 Introduction

Among the several known approaches to the theory of discrete positive systems, one can distinguish two basically different methods. The first one defines the state updating via a certain set of difference equations, involving positive matrices, and hence extracts summarizing features from the system trajectories. The second defines the admissible state transitions via a directed graph, and hence reduces the analysis of the above features to simple conditions on its connecting structure.
The former method is more powerful, as it provides answers to metric questions the latter is unable to deal with. There are problems, however, whose combinatorial nature is better enlightened when couched in graph theoretic terms. In this contribution we adopt this second viewpoint to discuss some aspects of positive homogeneous 2D linear systems [1].
The notion of 2-connectedness of a graph constitutes a natural starting point, and has the advantage of offering an easy introduction to imprimitivity classes, congruence relations etc., with which the interested reader is likely to be familiar in the one-dimensional context. The stress in the present approach is on the extension of classical Perron-Frobenius theory to matrix pairs and to their characteristic polynomials, and on the dynamical interpretation of some consequences of this theory. Moreover, the zero patterns of 2D system trajectories, arising either from a single initial condition or from an infinite set of initial conditions, are easily understood and described in algebraic terms when irreducible and primitive matrix pairs are involved.

Throughout the paper, given a matrix $F = [f_{ij}]$ (in particular, a vector), we write $F \gg 0$ (F *strictly positive*), if $f_{ij} > 0$ for all i, j; $F > 0$ (F *positive*), if $f_{ij} \geq 0$ for all i, j, and $f_{\bar{i}\bar{j}} > 0$ for some pair $(\bar{i}, \bar{j})$; $F \geq 0$ (F *nonnegative*), if $f_{ij} \geq 0$ for all i, j.

2 Paths and cycles in 2-digraphs

A *2-digraph* $\mathcal{D}^{(2)}$ is a triple $(V, \mathcal{A}, \mathcal{B})$, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of *vertices*, and $\mathcal{A}$ and $\mathcal{B}$ are subsets of $V \times V$ whose elements are called $\mathcal{A}$-*arcs* and $\mathcal{B}$-*arcs*, respectively. There is an $\mathcal{A}$-arc (a $\mathcal{B}$-arc) from v_i to v_j if (v_i, v_j) is in $\mathcal{A}$ (in $\mathcal{B}$).
If we denote by $\alpha(p)$ and $\beta(p)$ the number of $\mathcal{A}$-arcs and $\mathcal{B}$-arcs occurring in p, then $[\alpha(p) \quad \beta(p)]$ is the *composition* of p and $|p| = \alpha(p) + \beta(p)$ its *length.* A path whose extreme vertices coincide, i.e. $v_{i_0} = v_{i_k}$, is called a *cycle.* In particular, if each vertex appears exactly once as the first vertex of an arc, the cycle is called a *circuit.*

Definition A 2-digraph $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ is called
i) *strongly connected* if for every pair of vertices v_i and v_j in V there is a path p connecting v_i to v_j, i.e. $v_i \xrightarrow[p]{} v_j$;
ii) *2-strongly connected* if for every pair of vertices v_i and v_j in V there are two paths $v_i \xrightarrow[p_1]{} v_j$ and $v_i \xrightarrow[p_2]{} v_j$, connecting v_i to v_j, for which the ratios $\beta(p_i)/\alpha(p_i) \in \mathbb{R}_+ \cup \{+\infty\}$, $i = 1, 2$, are distinct.
As $\mathcal{D}^{(2)}$ is naturally associated with a 1-*digraph* (i.e. a standard digraph) $\mathcal{D}^{(1)} = (V, \mathcal{E})$, having the same vertices as $\mathcal{D}^{(2)}$ and $\mathcal{E} := \mathcal{A} \cup \mathcal{B}$ as its set of arcs, property i) corresponds to the fact that $\mathcal{D}^{(1)}$ is strongly connected, while property ii) is stronger as it constrains also the path compositions.
In the paper all 2-digraphs are assumed strongly connected with $\mathcal{A}, \mathcal{B} \neq \emptyset$.

We associate with the (finite) set $\{\gamma_1, \ldots, \gamma_t\}$ of all circuits in $\mathcal{D}^{(2)}$ (arbitrarily ordered) the *circuit matrix*

$$L(\mathcal{D}^{(2)}) := \begin{bmatrix} \alpha(\gamma_1) & \beta(\gamma_1) \\ \alpha(\gamma_2) & \beta(\gamma_2) \\ \vdots & \vdots \\ \alpha(\gamma_t) & \beta(\gamma_t) \end{bmatrix} \in \mathbb{N}^{t\times 2}, \qquad (1)$$

and denote by $M(\mathcal{D}^{(2)})$ the $\mathbb{Z}$-module generated by its rows. Since every cycle γ in $\mathcal{D}^{(2)}$ decomposes into a certain number of circuits, it follows that

$$[\alpha(\gamma)\ \ \beta(\gamma)] = [n_1\ \dots\ n_t]\ L(\mathcal{D}^{(2)}), \quad \exists\, n_1,\dots,n_t \in \mathbb{N}. \qquad (2)$$

As a submodule of $\mathbb{Z}^2$, $M(\mathcal{D}^{(2)})$ admits a basis consisting either of one or of two elements. In the first case $M(\mathcal{D}^{(2)})$ has only two possible bases, namely $\{[\ell\ m]\}$, for some positive integers ℓ and m, and its opposite $\{-[\ell\ m]\}$, and every circuit γ_j in $\mathcal{D}^{(2)}$ consists of $k_j\ell$ $\mathcal{A}$-arcs and k_jm $\mathcal{B}$-arcs, for a suitable k_j in $\mathbb{N}$. On the other hand, when $L(\mathcal{D}^{(2)})$ has rank 2, we can consider its Hermite form over $\mathbb{Z}$

$$\bar{H} := \begin{bmatrix} H \\ \hline 0 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ 0 & h_{22} \\ \hline \multicolumn{2}{c}{0} \end{bmatrix} = \bar{U} L(\mathcal{D}^{(2)}), \qquad (3)$$

$\bar{U} \in \mathbb{Z}^{t\times t}$ unimodular, and assume (without loss of generality) $h_{11}, h_{22} > 0$, and $0 \le h_{12} < h_{22}$. Then $\{[h_{11}\ h_{12}], [0\ h_{22}]\}$ represents a basis of $M(\mathcal{D}^{(2)})$, and the rows $[w_{11}\ w_{12}]$ and $[w_{21}\ w_{22}]$ of $W = UH$, as U varies over the group of unimodular matrices in $\mathbb{Z}^{2\times 2}$, give all possible bases of $M(\mathcal{D}^{(2)})$. Since all determinants $\det(UH)$ have the same modulus, which is a g.c.d. of the second order minors of $L(\mathcal{D}^{(2)})$, the parallelograms $\{\varepsilon[w_{11}\ w_{12}]+\delta[w_{21}\ w_{22}] : \varepsilon,\delta \in [0,1)\}$, have the same area, that coincides with the number of integer pairs they include [2, 7].

The cyclic structure of $\mathcal{D}^{(2)}$ and the module $M(\mathcal{D}^{(2)})$ provide enough information to decide whether the 2-digraph is 2-strongly connected, as shown in the following proposition, whose proof easily follows from the above discussion.

Proposition 2.1 *For a 2-digraph $\mathcal{D}^{(2)}$ the following facts are equivalent*
i) $\mathcal{D}^{(2)}$ is 2-strongly connected;
ii) there are two circuits γ_i and γ_j satisfying $\beta(\gamma_i)/\alpha(\gamma_i) \neq \beta(\gamma_j)/\alpha(\gamma_j)$;
iii) rank $L(\mathcal{D}^{(2)}) = 2$;
iv) $M(\mathcal{D}^{(2)})$ has a basis consisting of two elements.
■

As it is well-known, the lengths of all cycles in a strongly connected 1-digraph $\mathcal{D}^{(1)}$, with imprimitivity index h, are multiples of h, and there exists a positive integer T such that, for all integers $t \in [T, +\infty) \cap (h)$, there is a cycle in $\mathcal{D}^{(1)}$ of length t [9]. A similar statement holds for a 2-strongly connected digraph $\mathcal{D}^{(2)}$, upon considering for each cycle γ in $\mathcal{D}^{(2)}$ not just its length, but its composition $[\alpha(\gamma)\ \ \beta(\gamma)]$. In this case the module (h) and the half-line $[T, +\infty)$ have to be replaced by $M(\mathcal{D}^{(2)})$ and by a suitable convex cone in $\mathbb{R}^2_+$, respectively. To prove this fact, we need the following technical lemma, which extends a well-known result on the subsets of $\mathbb{N}$ [8].

Lemma 2.2 [6] *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{N}^2$, closed under addition, and M the $\mathbb{Z}$-module generated by $\mathcal{S}$. If $\mathcal{K}$ is the convex cone generated in $\mathbb{R}^2_+$ by the elements of $\mathcal{S}$, there exists $[u\ \ w] \in \mathcal{K}\cap M$ such that $\Big([u\ \ w] + \mathcal{K}\Big) \cap M \subseteq \mathcal{S}$.* ■

Proposition 2.3 *Let $\mathcal{S} := \{[\alpha(\gamma)\ \ \beta(\gamma)] \in \mathbb{N}^2 :$ γ a cycle in $\mathcal{D}^{(2)}\}$ be the set of compositions of all cycles of the 2-digraph $\mathcal{D}^{(2)}$.*
i) If $M(\mathcal{D}^{(2)})$ has rank 1 and is generated by $[\ell\ m] \in \mathbb{N}^2$, there exists $\tau \in \mathbb{N}$ s.t.

$$\{t\,[\ell\ \ m] : t \in \mathbb{N}, t \ge \tau\} \subseteq \mathcal{S} \subseteq \{t\,[\ell\ \ m] : t \in \mathbb{N}\}. \quad (4)$$

ii) If $M(\mathcal{D}^{(2)})$ has rank 2 and $\mathcal{K}$ denotes the solid convex cone generated by the rows of $L(\mathcal{D}^{(2)})$, there exists $[u\ \ w] \in M(\mathcal{D}^{(2)}) \cap \mathcal{K}$ such that

$$M(\mathcal{D}^{(2)}) \cap \Big([u\ \ w] + \mathcal{K}\Big) \subseteq \mathcal{S} \subseteq M(\mathcal{D}^{(2)}) \cap \mathcal{K}. \quad (5)$$

Proof Consider a cycle $\bar{\gamma}$ passing through all vertices of $\mathcal{D}^{(2)}$ and the set $\bar{\mathcal{S}}$ of compositions of all cycles having $\bar{\gamma}$ as a subcycle. As $\bar{\mathcal{S}}$ is an additively closed subset of $\mathcal{S}$ and generates $M(\mathcal{D}^{(2)})$, the lefthand inclusions in (4) and (5) follow from the previous lemma. The righthand inclusions are consequences of (2). ■

The vertices of a strongly connected 1-digraph $\mathcal{D} = (V, \mathcal{E})$, with imprimitivity index h, can be partitioned into h 1-*imprimitivity classes*, $C_1^{(1)}, C_2^{(1)}, \dots, C_h^{(1)}$. Any path moves from a starting vertex through all 1-imprimitivity classes, in a definite cyclic order, and returns to the class of the starting vertex after h arcs. We may index the classes so that any arc originated in $C_i^{(1)}$ enters $C_{i+1 \bmod h}^{(1)}$, and hence any path p originated in $C_i^{(1)}$ ends in $C_{i+|p| \bmod h}^{(1)}$. Also, if $|p|$ is large enough, the terminal vertex can be arbitrarily chosen within the class $C_{i+|p| \bmod h}^{(1)}$.

The above features extend to a strongly connected 2-digraph, provided that we look at the associated 1-digraph $\mathcal{D}^{(1)}$, and consider also the path compositions. In this case we generally obtain a finer partition of the set V, as every 1-imprimitivity class splits into a certain number of 2-*imprimitivity classes.*

Definition Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a 2-digraph. Two vertices v_i and $v_j \in V$ are said $\sim$-*equivalent* ($v_i \sim v_j$) if for every $v_k \in V$ there are paths $v_k \xrightarrow[p_{ik}]{} v_i$ and $v_k \xrightarrow[p_{jk}]{} v_j$ such that $[\alpha(p_{ik})\ \ \beta(p_{ik})] = [\alpha(p_{jk})\ \ \beta(p_{jk})]$.

This amounts to say that it is possible to connect each vertex of V to v_i and v_j by resorting to two paths with the same composition. The equivalence relation $\sim$ induces a partition of V into disjoint 2-imprimitivity classes, whose number is called *2-imprimitivity index* and denoted by $h^{(2)}$. As paths with the same composition have the same length, $v_i \sim v_j$ implies that v_i and v_j belong to the same $C_\nu^{(1)}$, thus showing that every 2-imprimitivity class is a subset of a 1-imprimitivity class.

Lemma 2.4 [6] *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a 2-digraph. For every pair of vertices v_i and v_j in V, there exist $\alpha_{ji}, \beta_{ji} \in \mathbb{N}$ such that any path $v_i \xrightarrow[p]{} v_j$ satisfies*

$$[\alpha(p) \;\; \beta(p)] \equiv [\alpha_{ji} \;\; \beta_{ji}] \qquad \bmod M(\mathcal{D}^{(2)}). \qquad (6)$$

In particular, if v_i and v_j belong to the same $\sim$-equivalence class, (6) holds for $[\alpha_{ji} \;\; \beta_{ji}] = [0 \;\; 0]$. Finally, if $v_\ell \sim v_i$ and $v_m \sim v_j$, condition (6) holds for any path connecting v_ℓ to v_m. ■

As a consequence of (6), once a particular 2-imprimitivity class has been selected as a reference, all classes can be unambiguously indexed by the elements of the quotient module $\mathbb{Z}^2/M(\mathcal{D}^{(2)})$, in the sense that each class is indexed by a coset $[\alpha(p) \;\; \beta(p)] + M(\mathcal{D}^{(2)})$, p being any path that reaches the class, starting from the reference one. We may ask under what conditions the above correspondence, mapping 2-imprimitivity classes into cosets, is bijective. Clearly, when $M(\mathcal{D}^{(2)})$ has rank 1, the quotient module $\mathbb{Z}^2/M(\mathcal{D}^{(2)})$ includes infinitely many elements, and no bijection exists between the (finite) set of 2-imprimitivity classes and $\mathbb{Z}^2/M(\mathcal{D}^{(2)})$.
On the other hand, when the module $M(\mathcal{D}^{(2)})$ has rank 2, this correspondence always exists, as shown in the following proposition.

Proposition 2.5 *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a 2-digraph and $\mathcal{K}$ the solid convex cone generated in $\mathbb{R}_+^2$ by the rows of $L(\mathcal{D}^{(2)})$. For every integer pair $[h \;\; k]$ in $\mathcal{K}$ there are vertices v_i and v_j and a path $v_i \xrightarrow[p]{} v_j$ s.t. $[\alpha(p) \;\; \beta(p)] = [h \;\; k]$.*

Proof Possibly after reordering the rows of $L(\mathcal{D}^{(2)})$, we can assume that the ratios $\beta(\gamma_i)/\alpha(\gamma_i) \in \mathbb{R}_+ \cup \{+\infty\}$ satisfy $\beta(\gamma_i)/\alpha(\gamma_i) \le \beta(\gamma_{i+1})/\alpha(\gamma_{i+1})$, $i = 1, 2, \dots, t-1$. So, $[\alpha(\gamma_1) \;\; \beta(\gamma_1)]$ and $[\alpha(\gamma_t) \;\; \beta(\gamma_t)]$ determine the extremal rays of $\mathcal{K}$, and for every $[h \;\; k] \in \mathcal{K}$ we have $\beta(\gamma_1)/\alpha(\gamma_1) \le k/h \le \beta(\gamma_t)/\alpha(\gamma_t)$. Consider some $[h \;\; k] \in \mathcal{K} \cap \mathbb{N}^2$ and a path $p_0 = (v_{i_0}, v_{i_1}), (v_{i_1}, v_{i_2}), \dots, (v_{i_{h+k-1}}, v_{i_{h+k}})$ of length $h+k$. If $\alpha(p_0) = h$, and hence $\beta(p_0) = k$, we are done; if not, suppose, for instance, $\alpha(p_0) > h$. We can first extend p_0 into a cycle $\tilde{\gamma}$, passing through some vertices v_p of γ_1 and v_q of γ_t, arbitrarily selected, and then extend $\tilde{\gamma}$ into a new cycle γ, by adding n_1 copies of circuit γ_1 and n_t copies of γ_t. We can select n_1 and n_t so that

$$\frac{n_1\beta(\gamma_1) + n_t\beta(\gamma_t) + \beta(\tilde{\gamma})}{n_1\alpha(\gamma_1) + n_t\alpha(\gamma_t) + \alpha(\tilde{\gamma})} \ge \frac{k}{h} \ge \frac{\beta(\gamma_1)}{\alpha(\gamma_1)}. \qquad (7)$$

Set $N := n_1\Big(\alpha(\gamma_1) + \beta(\gamma_1)\Big) + n_t\Big(\alpha(\gamma_t) + \beta(\gamma_t)\Big) + \Big(\alpha(\tilde{\gamma}) + \beta(\tilde{\gamma})\Big)$ and $\gamma = (v_{i_0}, v_{i_1}), \dots, (v_{i_{h+k-1}}, v_{i_{h+k}}), \dots, (v_{i_{N-1}}, v_{i_0})$. Consider the family of all paths of length $h+k$ described as $p_r = (v_{i_r}, v_{i_{r+1}}), \dots, (v_{i_{r+h+k-1 \bmod N}}, v_{i_{r+h+k \bmod N}})$, $r = 0, 1, \dots, N-1$. As $|\alpha(p_r) - \alpha(p_{r+1})| \le 1$ for every r, either the family includes a path with h $\mathcal{A}$-arcs, and the proof is complete, or all paths p_r have $\alpha(p_r) > h$ $\mathcal{A}$-arcs, and hence $\sum_{r=0}^{N-1} \alpha(p_r) > hN$. As in γ there are $n_1\alpha(\gamma_1) + n_t\alpha(\gamma_t) + \alpha(\tilde{\gamma})$ $\mathcal{A}$-arcs, each of them belonging to $h+k$ different paths p_r, it follows that

$$\sum_{r=0}^{N-1} \alpha(p_r) = \Big(n_1\alpha(\gamma_1) + n_t\alpha(\gamma_t) + \alpha(\tilde{\gamma})\Big)(h+k).$$

So, $\alpha(p_r) > h$ for all paths p_r implies $\Big(n_1\alpha(\gamma_1) + n_t\alpha(\gamma_t) + \alpha(\tilde{\gamma})\Big)(h+k) > hN$, and therefore $\Big(n_1\alpha(\gamma_1) + n_t\alpha(\gamma_t) + \alpha(\tilde{\gamma})\Big)k > \Big(n_1\beta(\gamma_1) + n_t\beta(\gamma_t) + \beta(\tilde{\gamma})\Big)h$, which contradicts (7). ■

Corollary 2.6 *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a 2-digraph. If $M(\mathcal{D}^{(2)})$ has rank 2, for every $[h \;\; k] \in \mathbb{N}^2$ there are vertices v_i and v_j and a path $v_i \xrightarrow[p]{} v_j$ s.t.*

$$[\alpha(p) \;\; \beta(p)] \equiv [h \;\; k] \qquad \bmod M(\mathcal{D}^{(2)}). \qquad (8)$$

Proof If $[h \;\; k]$ belongs to the convex cone $\mathcal{K}$, generated in $\mathbb{R}_+^2$ by the rows of $L(\mathcal{D}^{(2)})$, then (8) holds by Proposition 2.5. If not, by exploiting the fact that $\mathcal{K}$ is a solid cone, we can find $n_1, n_2, \dots, n_t \in \mathbb{N}$, such that $[\tilde{h} \;\; \tilde{k}] := [h \;\; k] + [n_1 \;\; n_2 \;\; \dots \;\; n_t]\, L(\mathcal{D}^{(2)})$ is an element of $\mathcal{K}$. By the above proposition, there are vertices v_i and v_j and a path $v_i \xrightarrow[p]{} v_j$ such that $[\alpha(p) \;\; \beta(p)] = [\tilde{h} \;\; \tilde{k}]$, and hence (8) holds true. ■

When $M(\mathcal{D}^{(2)})$ has rank 2 and a reference class has been selected, there is a bijection between 2-imprimitivity classes and cosets of $\mathbb{Z}^2/M(\mathcal{D}^{(2)})$. Therefore, the number of 2-imprimitivity classes is $h^{(2)}$ and coincides with the number of integer pairs included in the parallelogram $\{\varepsilon\, [w_{11} \; w_{12}] + \delta\, [w_{21} \; w_{22}] : \varepsilon, \delta \in [0, 1)\}$, for every basis $\{[w_{11} \; w_{12}], [w_{21} \; w_{22}]\}$ of $M(\mathcal{D}^{(2)})$. For instance, if we refer to the basis of $M(\mathcal{D}^{(2)})$ obtained from the Hermite form of $L(\mathcal{D}^{(2)})$, we can index the 2-imprimitivity classes on the set

$$\mathcal{I} \;:=\; \{[i \;\; j] \in \mathbb{N}^2 : [i \;\; j] = \varepsilon\, [h_{11} \; h_{12}] + \delta\, [0 \; h_{22}], \; \exists\, \varepsilon, \delta \in [0, 1)\}$$

and denote each of them as $C_{ij}^{(2)}$, $[i \;\; j] \in \mathcal{I}$. If $C_{00}^{(2)}$ denotes the reference class, all paths from vertices in

$C^{(2)}_{00}$ to vertices in $C^{(2)}_{ij}$ have compositions congruent to $[i \;\; j]$ mod $M(\mathcal{D}^{(2)})$, and hence lengths congruent to $i+j$ mod h (the imprimitivity index of $\mathcal{D}^{(1)}$). This implies that $C^{(2)}_{ij}$ and $C^{(2)}_{00}$ are included in the same 1-imprimitivity class if and only if $i+j \equiv 0$ mod h.

Proposition 2.7 *Let $\mathcal{D}^{(2)} = (V, \mathcal{A}, \mathcal{B})$ be a 2-digraph, with rank $M(\mathcal{D}^{(2)}) = 2$, and denote by h and $h^{(2)}$ its 1- and 2-imprimitivity indices, respectively. All 1-imprimitivity classes of $\mathcal{D}^{(2)}$ include the same number q of 2-imprimitivity classes, so that $h^{(2)} = qh$.*

PROOF Let $C^{(1)}_\nu$ be an arbitrary 1-imprimitivity class. Select any 2-imprimitivity class included in $C^{(1)}_\nu$ as a reference, and denote it by $C^{(2)}_{00}$. As $C^{(2)}_{ij} \subseteq C^{(1)}_\nu$ if and only if $i+j \equiv 0$ mod h, the number of 2-imprimitivity classes included in $C^{(1)}_\nu$ coincides with that of the integer pairs $[i \;\; j]$ in $\mathcal{I}$ satisfying $h \mid (i+j)$. Since this number is independent of both the particular $C^{(1)}_\nu$ and the 2-imprimitivity class selected in it, the result holds true. ∎

3 Irreducible pairs and 2D systems

Given a nonnegative matrix $F = [f_{ij}] \in \mathbb{R}^{n\times n}_+$, it is possible to associate it with an essentially unique 1-digraph, $\mathcal{D}^{(1)}(F)$, with vertices, $v_1, v_2, \dots, v_n$. There is an arc from v_i to v_j if and only if $f_{ji} > 0$. This correspondence extends to matrix pairs, as we can associate with every pair (A, B) of $n \times n$ nonnegative matrices a 2-digraph $\mathcal{D}^{(2)}(A, B)$ with vertices, $v_1, v_2, \dots, v_n$. There is an $\mathcal{A}$-arc (a $\mathcal{B}$-arc) from v_j to v_i if and only if the (i, j)th entry of A (of B) is nonzero [1].

The combinatorial properties of a pair (A, B) with a 2-strongly connected digraph appear as natural generalizations of those of an irreducible matrix, i.e. a matrix with a strongly connected digraph. Indeed, the dynamical behavior of the 2D state model described by (A, B) eventually exhibits a two-dimensional periodic pattern, and the "extremal" zeros of its characteristic polynomial are periodically distributed on a torus. This motivates the following definition.

Definition A pair (A, B) of $n \times n$ positive matrices is *irreducible* if $\mathcal{D}^{(2)}(A, B)$ is 2-strongly connected.

Notice that this amounts to require that $A + B$ is irreducible and $L_{A,B}$ has rank 2. So, in particular, all pairs (A, B) with $A+B$ primitive are irreducible, but the converse is not true. The irreducibility of a matrix pair (A, B) can be characterized by referring to the dynamical behavior of the associated *2D system* [3]

$$\mathbf{x}(h+1, k+1) = A\mathbf{x}(h, k+1) + B\mathbf{x}(h+1, k), \quad (9)$$

$h, k \in \mathbb{Z}$, $h + k \geq 0$, where the *local states* $\mathbf{x}(h, k)$ are elements of $\mathbb{R}^n_+$ and *initial conditions* are given by assigning a sequence $\mathcal{X}_0 := \{\mathbf{x}(\ell, -\ell) : \ell \in \mathbb{Z}\}$ of nonnegative local states on the *separation set* $\mathcal{S}_0 := \{(\ell, -\ell) : \ell \in \mathbb{Z}\}$. If the initial conditions on $\mathcal{S}_0$ are all zero, except at $(0, 0)$, we have

$$\mathbf{x}(h, k) = (A^h \sqcup^k B)\, \mathbf{x}(0, 0), \quad \forall\, h, k \in \mathbb{N},$$

where the *Hurwitz products* $A^h \sqcup^k B$ of A and B are inductively defined [4] as

$$A^h \sqcup^0 B = A^h, \;\; h \geq 0, \quad \text{and} \quad A^0 \sqcup^k B = B^k, \;\; k \geq 0, \quad (10)$$

and, when h and k are both positive,

$$A^h \sqcup^k B = A(A^{h-1} \sqcup^k B) + B(A^h \sqcup^{k-1} B). \quad (11)$$

The following lemma shows that for an irreducible pair (A, B) the Hurwitz products belonging to a finite window $\mathcal{F}$ sum up to a strictly positive matrix when $\mathcal{F}$ is large enough and moves in a suitable cone of $\mathbb{R}^2_+$.

Lemma 3.1 [6] *A pair of $n \times n$ positive matrices (A, B) is irreducible if and only if there are a solid convex cone $\mathcal{K}^*$ and a finite set $\mathcal{F} \subset \mathbb{N}^2$ s.t.*

$$\sum_{[r \;\; s] \in [h \;\; k] + \mathcal{F}} A^r \sqcup^s B \gg 0, \quad (12)$$

$\forall\, [h \;\; k] \in \mathbb{N}^2$ s.t. $[h \;\; k] + \mathcal{F} \subset \mathcal{K}^*$. ∎

If $\mathcal{X}_0$ consists of a single nonzero local state at $(0, 0)$, condition (12) can be restated as

$$\sum_{[i \;\; j] \in [h \;\; k] + \mathcal{F}} \mathbf{x}(i, j) \gg 0, \quad (13)$$

for every pair $[h \;\; k] \in \mathbb{N}^2$ s.t. $[h \;\; k] + \mathcal{F} \subset \mathcal{K}^*$. When there is an infinite number of nonzero local states on $\mathcal{S}_0$, the state evolution possibly affects the whole half-plane $\{(h, k) \in \mathbb{Z}^2 : h + k \geq 0\}$. We may ask whether there is a separation set $\mathcal{S}_\nu = \{(h, k) \in \mathbb{Z}^2 : h + k = \nu\}$ such that (13) is fulfilled by all pairs $[h \;\; k]$ beyond $\mathcal{S}_\nu$, i.e. satisfying $h + k \geq \nu$.
This is clearly impossible if no upper bound exists on the distance between consecutive nonzero local states on $\mathcal{S}_0$. If we confine ourselves to *admissible* sets of initial conditions, namely to nonnegative sequences $\mathcal{X}_0$ for which *there is an integer $N > 0$ such that* $\sum_{\ell=h}^{h+N} \mathbf{x}(\ell, -\ell) > 0$ *for all $h \in \mathbb{Z}$*, irreducibility can be characterized as follows.

Proposition 3.2 *A pair (A, B) of $n \times n$ positive matrices is irreducible if and only if there is a finite set $\mathcal{F} \subset \mathbb{N}^2$ such that for every admissible set of initial*

[1] As in the sequel we always refer to the 2-digraph $\mathcal{D}^{(2)}(A, B)$, associated with a specific matrix pair (A, B), we denote the circuit matrix $L(\mathcal{D}^{(2)}(A, B))$ by $L_{A,B}$ and the corresponding module by $M_{A,B}$.

conditions $\mathcal{X}_0$ a positive integer T can be found such that

$$\sum_{[i\ \ j]\in[h\ \ k]+\mathcal{F}} \mathbf{x}(i,j) \gg 0, \quad \forall\ [h\ \ k] \in \mathbb{Z}^2 \text{ s.t. } h+k \geq T. \tag{14}$$

Proof Assume (A, B) irreducible and let $\mathcal{K}^*$ be the solid convex cone and $\mathcal{F}$ the finite set considered in Lemma 3.1. If $\mathbf{x}(\ell, -\ell)$ is a nonzero state belonging to an admissible $\mathcal{X}_0$, (13) holds for every $[h\ \ k] \in \mathbb{Z}^2$ such that $[h\ \ k]+\mathcal{F} \subseteq [\ell\ \ -\ell]+\mathcal{K}^*$. As the distance between consecutive nonzero states on $\mathcal{S}_0$ is upper bounded, the union of all solid convex cones $[\ell\ \ -\ell] + \mathcal{K}^*$, for $[\ell\ \ -\ell]$ varying over the support of $\mathcal{X}_0$, includes the half-plane $\{[h\ \ k] \in \mathbb{Z}^2 : h + k \geq T\}$ for T large enough, which proves the statement.
Conversely, suppose that (A, B) is not irreducible. If $A+B$ is reducible, we can assume, possibly by resorting to a cogredience transformation,

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ 0 & A_{22} + B_{22} \end{bmatrix},$$

$A_{11}, B_{11} \in \mathbb{R}_+^{r\times r},\ 0 < r < n$. Then, for any admissible set of initial conditions $\mathcal{X}_0$ whose local state vectors have the last $n - r$ entries identically zero, condition (14) is not fulfilled.
On the other hand, if $A + B$ is irreducible and rank $L_{A,B} = 1$, there is a strip $\mathcal{H} \subset \mathbb{N}^2$, including all pairs $[h\ \ k]$ corresponding to some paths in $\mathcal{D}^{(2)}(A, B)$, and hence to nonzero Hurwitz products. Consequently, admissible sets of initial conditions can be found such that two strips $[\ell\ \ -\ell] + \mathcal{H}$, corresponding to consecutive nonzero local states $\mathbf{x}(\ell, -\ell)$, do not intersect. For this kind of admissible sets (14) is not verified. ■

Up to this point, we have considered a nonnegative matrix pair (A, B) only from the point of view of its incidence graph. Important tools for analysing the properties of (A, B) are its *characteristic polynomial*, defined as

$$\begin{aligned} \Delta_{A,B}(z_1, z_2) &:= \det(I_n - Az_1 - Bz_2) \\ &= \textstyle\sum_{h,k\in\mathbb{N}} d_{hk} z_1^h z_2^k, \quad d_{00} = 1, \end{aligned}$$

and the associated *variety* $\mathcal{V}(\Delta_{A,B})$, i.e. the set of points $(\lambda, \mu) \in \mathbb{C}^2$ such that $\det(I_n - A\lambda - B\mu) = 0$. We aim to discuss certain connections between $\operatorname{supp}(\Delta_{A,B})$ and the circuit matrix $L_{A,B}$, and to show that the *support matrix*

$$S_{A,B} := \begin{bmatrix} h_1 & k_1 \\ \vdots & \vdots \\ h_r & k_r \end{bmatrix} \tag{15}$$

and $L_{A,B}$ provide the same information about the irreducibility of (A, B). This approach is intimately connected with the classical Perron-Frobenius theory for a single positive matrix, and suggests the possibility of obtaining a description of irreducible pairs in terms of the associated characteristic polynomials. The key results we need are summarized in the next couple of lemmas, whose proof is rather technical and can be found in [6].

Lemma 3.3 *Let A and B be positive matrices with $A + B$ irreducible and $\rho(A + B) = r$. For any θ and $\omega \in \mathbb{R}$ the following facts are equivalent:*
i) $(r^{-1}e^{i\theta}, r^{-1}e^{i\omega})$ belongs to $\mathcal{V}(\Delta_{A,B})$;
ii) for every cycle γ in $\mathcal{D}^{(2)}(A, B)$, including $\alpha(\gamma)$ $\mathcal{A}$-arcs and $\beta(\gamma)$ $\mathcal{B}$-arcs,

$$\alpha(\gamma)\theta + \beta(\gamma)\omega \equiv 0 \qquad \text{mod } 2\pi; \tag{16}$$

iii) the characteristic polynomial satisfies $\Delta_{A,B}(z_1, z_2) = \Delta_{A,B}(z_1 e^{i\theta}, z_2 e^{i\omega})$;
iv) $h\theta + k\omega \equiv 0 \text{ mod } 2\pi$ for every $(h, k) \in \operatorname{supp}(\Delta_{A,B})$. ■

Lemma 3.4 [6] *Let L and S be arbitrary integer matrices with the same number n of columns. The two congruences*

$$L\Theta \equiv 0 \text{ mod } \mathbb{Z}, \quad S\Theta \equiv 0 \text{ mod } \mathbb{Z}, \quad \Theta \in \mathbb{Q}^n, \tag{17}$$

have the same set of solutions if and only if the $\mathbb{Z}$-modules M_L and M_S, generated by the rows of L and by the rows of S, respectively, coincide. ■

Proposition 3.5 *Let (A, B) be a pair of $n \times n$ positive matrices, with $A+B$ irreducible. The $\mathbb{Z}$-modules generated by the rows of $L_{A,B}$ and by the rows of $S_{A,B}$ coincide.*

Proof As $A + B$ is irreducible, condition (16) holds for every cycle γ if and only if

$$L_{A,B} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv 0 \quad \text{mod } 2\pi. \tag{18}$$

So, by Proposition 4.1, the congruence $S_{A,B} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv 0 \text{ mod } 2\pi$ and that in (18) have the same sets of solutions, and the result is a direct consequence of the previous lemma, upon replacing $[\theta\ \ \omega]^T$ with $\Theta := [\theta/2\pi\ \ \omega/2\pi]^T$. ■

Perron-Frobenius theorem undoubtely constitutes the most significative result about irreducible matrices, as it clarifies their spectral structure and provides useful information on the asymptotic behavior of the associated state models. Interestingly enough, the varieties of irreducible matrix pairs exhibit features that appear as natural extensions of the properties enlightened by Perron-Frobenius theorem, a result that further corroborates the definition of irreducibility.

Proposition 3.6 [6] [2D Perron-Frobenius theorem] *Let (A, B) be an irreducible pair of $n \times n$ positive matrices, with $\rho(A + B) = r$. The variety $\mathcal{V}(\Delta_{A,B})$ intersects the polydisc $\mathcal{P}_{r^{-1}} := \{(z_1, z_2) \in$*

$\mathbb{C}^2 : |z_1| \le r^{-1}, |z_2| \le r^{-1}\}$ *only in* (r^{-1}, r^{-1}), *and in a finite number of points of its distinguished boundary* $\mathcal{T}_{r^{-1}} := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = r^{-1}, |z_2| = r^{-1}\}$. *Moreover,* (r^{-1}, r^{-1}) *is a regular point of the variety and there exists* $\mathbf{w} \gg 0$ *s.t.*

$$\left(I_n - r^{-1}A - r^{-1}B\right) \mathbf{w} = \mathbf{0}. \quad \blacksquare \qquad (19)$$

Corollary 3.7 *Let* (A, B) *be an irreducible pair of* $n \times n$ *positive matrices with* $\rho(A + B) = r$, *and let*

$$\bar{H} := \begin{bmatrix} H \\ \hline 0 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ 0 & h_{22} \\ \hline \multicolumn{2}{c}{0} \end{bmatrix} \in \mathbb{N}^{t \times 2}$$

be the Hermite form of $L_{A,B}$. *The variety of* $\Delta_{A,B}(z_1, z_2)$ *intersects* $\mathcal{P}_{r^{-1}}$ *exactly in the points* $(r^{-1}e^{i\theta}, r^{-1}e^{i\omega})$, *one obtains by varying* (θ, ω) *in the set*

$$\left\{ \left(\alpha\frac{2\pi}{h_{11}} + \beta\frac{2\pi h_{12}}{h_{11}h_{22}},\ \beta\frac{2\pi}{h_{22}} \right); \alpha, \beta \in \mathbb{N} \right\}. \qquad (20)$$

Consequently, the cardinality of $\mathcal{V}(\Delta_{A,B}) \cap \mathcal{P}_{r^{-1}}$ *coincides with the imprimitivity index of the pair.*

Proof As $\bar{H} = UL_{A,B}$ for a suitable unimodular matrix $U \in \mathbb{Z}^{t \times t}$, the congruences $H \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv \mathbf{0}$ and $L_{A,B} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \equiv \mathbf{0}$ modulo 2π, have the same sets of solutions which, by Lemma 3.1, coincide with $\mathcal{V}(\Delta_{A,B}) \cap \mathcal{P}_{r^{-1}}$. It is easy to verify that the distinct pairs (modulo 2π) in (20) amounts to $h_{11}h_{22} = \det H = h^{(2)}$. $\blacksquare$

4 Concluding remark

In [5] the notion of primitivity for a positive pair (A, B) was introduced as a strict positivity constraint on the asymptotic dynamics of the associated 2D state model.

In this framework primitivity can be viewed as a special case of irreducibility: an irreducible pair is called *primitive* if its imprimitivity index $h^{(2)}$ is 1. As an immediate corollary of the previous propositions, we have that (A, B) is primitive if and only if any of the following (equivalent) conditions holds:

i) $L_{A,B}$ is a primitive matrix;

ii) $S_{A,B}$ is a primitive matrix;

iii) $M_{A,B}$ coincides with $\mathbb{Z}^2$;

iv) there exists a strictly positive Hurwitz product;

v) there is a solid convex cone $\mathcal{K}^*$ in $\mathbb{R}^2_+$ such that for all $(h, k) \in \mathbb{N}^2 \cap \mathcal{K}^*$ the Hurwitz product $A^h \sqcup\!\sqcup^k B$ is strictly positive;

vi) for every admissible set of initial conditions there is a positive integer T such that $\mathbf{x}(h, k) \gg 0$ for all $(h, k) \in \mathbb{N}^2$, $h + k \ge 0$;

vii) the variety $\mathcal{V}(\Delta_{A,B})$ intersects the polydisk $\mathcal{P}_{r^{-1}}$ only in (r^{-1}, r^{-1}).

References

[1] N.K. Bose , *Multidimensional Systems Theory* , D.Reidel Publ. Co., Dordrecht (NL) , 1985

[2] J.W. Cassels, *An introduction to the geometry of numbers* , Springer, Berlin, 1959

[3] E. Fornasini and G. Marchesini, “ Doubly indexed dynamical systems ”, *Math.Sys. Theory* , vol.12, pp. 59-72, 1978

[4] E. Fornasini and M.E. Valcher, “Matrix pairs in 2D systems: an approach based on trace series and Hankel matrices ”, *SIAM J. of Contr. and Optim.*, vol. 33, pp. 1127-1150, 1995

[5] E. Fornasini and M.E. Valcher, “Primitivity of positive matrix pairs: algebraic characterization, graph-theoretic description and 2D systems interpretation ”, submitted 1995

[6] E. Fornasini and M.E. Valcher, “Directed graphs, 2D state models and characteristic polynomial of irreducible matrix pairs ”, submitted 1996

[7] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, Oxford Science Publ., 1979

[8] J.G. Kemeny and J.L. Snell, *Finite Markov chains* , Van Nostrand Reinhold Co., NY , 1960

[9] H. Minc , *Nonnegative Matrices* , J.Wiley & Sons, New York , 1988