STRUCTURAL PROPERTIES OF 2D COMPARTMENTAL MODELS

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ABSTRACT Two-dimensional (2D) compartmental models are 2D positive systems obeying some conservation law, and hence described by matrix pairs with substochastic sum. A canonical form, to which all 2D compartmental models reduce, is derived, allowing for a complete analysis of stability properties. The relevance of these models is illustrated by two examples: the single-carriageway traffic flow and the Streeter-Phelps discrete model.

KEYWORDS: system modelling, 2D positive and compartmental models, stability

1 INTRODUCTION

During the last decades compartmental modelling techniques have been increasingly applied to the analysis of biological and chemical processes, and, in general, for investigating dynamical systems to which some law of conservation (of matter, of energy, etc.) applies [4]. Typically, compartmental models consist of a finite number of compartments with specified interconnections that either represent fluxes of materials from one site to another or chemical transformations or both. Consequently, their behavior is described by a set of ordinary difference or differential equations. Sometimes, however, the phaenomenon one aims to model is intrinsically multidimensional, as both time and spatial coordinates are involved, and one resorts to PDE's or multidimensional (nD) discrete systems.

2D compartmental models are 2D positive systems [3, 6] endowed with the property that the matrix pair responsible for the state updating has a substocastic sum. This constraint entails far reaching consequences on the stability properties, and allows to derive a canonical form for 2D compartmental models which gives deep insights into their asymptotic behavior.

In order to understand the significance of 2D compartmental systems, it will be useful to have a couple of applications in mind as examples of the phaenomena we are trying to describe. In both cases, the derivation of the model requires many simplifying assumptions, and 2D difference equations we obtain provide only a crude account of traffic flow and river selfpurification. We will concentrate, instead, on some aspects that illustrate how these examples can be viewed as paradigms of a broad class of dynamical behaviors, that can be investigated by applying 2D compartmental systems techniques.

Example 1 [SINGLE-CARRIAGEWAY TRAFFIC FLOW] Our aim is to represent, by means of a discrete model, the traffic flow along one carriageway of a motorway. To this end we introduce the following assumptions:

a) The road is partitioned into stretches of length L and the time into intervals of duration T.

b) At time instant $tT, t \in \mathbb{Z}$, the set of cars inside the stretch $[\ell L, (\ell+1)L), \ell \in \mathbb{Z}$, is partitioned into groups of equal speed span, say V m.p.h. This amounts to saying that the first group consists of all cars whose speed belongs to the interval $(0, V]$, in the second group there are all cars with speed in $(V, 2V]$, ... Also, one more group is considered, including all cars that at time tT are temporarily stopping (at a gas station, in a parking place, ...). The groups are sequentially indexed from 0 through n, with 0 denoting the class of stopping cars, 1 the lowest speed group and n the highest. If $v_i(\cdot,\cdot)$ represents the number of cars belonging to the i-th group, the "state" at time tT of the ℓ-th stretch, $[\ell L, (\ell+1)L)$, is given by $\mathbf{v}(\ell,t) = [\, v_0(\ell,t) \quad \dots \quad v_n(\ell,t)\,]^T$.

c) The number of vehicles is large enough to assume that the v_i's are continuous variables.

d) Inputs and outputs at motorway intersections are modelled apart. Typically, only some stretches exhibit an intersection, and it is obvious that the output levels in $[tT, (t+1)T)$ cannot exceed the number of cars running through those stretches in that time interval.

e) Car drivers belonging to the i-th group at time tT exhibit a propension p_{ji} (probability) to istantaneously move to the j-th speed class at the beginning of the next time interval, and to drive at that speed during $(tT, (t+1)T]$. Clearly, $\sum_{j=0}^n p_{ji} = 1$.

f) The length L of a road stretch satisfies $L > nVT$. Every car in the ℓ-th stretch at time tT, at time $(t+1)T$ belongs either to the same stretch or to the $(\ell+1)$-th. If there are r cars moving within the i-th speed class in the time interval $[tT, (t+1)T)$, uniformly distributed at time tT along the stretch $[\ell L, (\ell+1)L)$, only $g_i r$ of them, with $g_i := \frac{(2i-1)VT}{2L}$, reach the next stretch before $(t+1)T$. The remaining $(1-g_i)r$ cars are still in $[\ell L, (\ell+1)L)$ at time $(t+1)T$.

If we disregard outflows and inflows, we get the following model:

$$\mathbf{v}(\ell+1, t+1) = GP\mathbf{v}(\ell,t) + (I_{n+1} - G)P\mathbf{v}(\ell+1,t), \quad (1)$$

with $G = \text{diag}\{0, g_1, \dots, g_n\}$ and $P = [p_{ij}]_{ij}$, and, by resorting to the transformation $\mathcal{T} : \mathbb{Z}^2 \to \mathbb{Z}^2 : (\ell, t) \mapsto (h, k) = (\ell, t - \ell)$ and assuming $\mathbf{x}(h, k) := \mathbf{v}\Big(\mathcal{T}^{-1}(h, k)\Big) = \mathbf{v}(h, h + k)$, we can rewrite (1) as

$$\mathbf{x}(h+1, k+1) = GP\mathbf{x}(h, k+1) + (I_{n+1} - G)P\mathbf{x}(h+1, k). \quad (2)$$

Example 2 [Streeter-Phelps discrete model] [1] In modelling the self-purification process of a polluted river, we introduce the following assumptions:

a) The variety of pollutants inside the river reduces to one class of oxidizable substances, whose concentration is measured by the amount of oxygen (BOD = biological oxygen demand) needed for their oxidation.

b) Selfpurification is due to dissolved oxygen (DO) which oxidizes polluting materials and eventually convert them into abiotic substances and heat.

c) As the variations of BOD and DO concentrations on river cross sections are less significative than the longitudinal ones, we assume a (spatially) one-dimensional model. Moreover, idrological variables and, in particular, the stream velocity V, are constant all over the river.

d) The river is divided into reaches of length L. The time step T is given by $T = \frac{L}{V}$.

We denote by $\beta(\ell, t)$ and $\delta(\ell, t)$ the concentration of BOD and the deficit of DO w.r.t. the saturation level in the ℓth reach at time tT. BOD and DO at $\big((\ell + 1)L, (t + 1)T\big)$ are obtained on the basis of a discretized balance equation accounting for different contributions.

- *Diffusion* is modelled by assuming that the BOD content of the water volume centered in ℓL at time tT undergoes in $[tT, (t + 1)T)$ a variation proportional to the differences $\beta(\ell-1, t) - \beta(\ell, t)$ and $\beta(\ell+1, t) - \beta(\ell, t)$. Same assumption is made for the DO.
- *Self-purification*: in the time interval $[tT, (t + 1)T)$ the BOD concentration in the ℓ-th river reach is decreased by the same amount $a_1 T\beta(\ell, t)$ the DO deficit is increased.
- *Reaeration* takes place at the water-atmosphere interface. We assume that in $[tT, (t + 1)T)$ the DO deficit is reduced of an amount given by $a_2 T\delta(\ell, t)$.
- *BOD sources*: effluents, local run-off, etc., modifying the BOD concentration, determine an exogenous input to the system, which is denoted by $\mathbf{u}_\beta(\cdot, \cdot)$.

By making the above assumptions, we obtain:

$$\begin{aligned}\begin{bmatrix}\beta(\ell+1, t+1)\\ \delta(\ell+1, t+1)\end{bmatrix} &= S\begin{bmatrix}\beta(\ell, t)\\ \delta(\ell, t)\end{bmatrix} + D\begin{bmatrix}\beta(\ell-1, t)\\ \delta(\ell-1, t)\end{bmatrix}\\ +D\begin{bmatrix}\beta(\ell+1, t)\\ \delta(\ell+1, t)\end{bmatrix} &+ \begin{bmatrix}\tilde{M}\\ 0\end{bmatrix}\mathbf{u}_\beta(\ell, t), \end{aligned}\quad (3)$$

where

$$\begin{aligned} S &= [s_{ij}]_{ij} = \begin{bmatrix}1 - a_1T - 2D_\beta T & 0\\ a_1 T & 1 - a_2 T - 2D_\delta T\end{bmatrix}\\ D &= [d_{ij}]_{ij} = \begin{bmatrix}D_\beta T & 0\\ 0 & D_\delta T\end{bmatrix}.\end{aligned}$$

As $\tilde{M}, a_1, a_2, D_\beta$ and D_δ are positive and T is small, all matrices in the above equation are positive. The model (3) can be reduced to an equivalent one having the structure (2). Actually, upon defining

$$\mathbf{z}(\ell, t) := \left[\begin{array}{c}\beta(2\ell, t)\\ \beta(2\ell+1, t)\\ \hline \delta(2\ell, t)\\ \delta(2\ell+1, t)\end{array}\right] \quad \text{and} \quad \tilde{\mathbf{u}}(\ell, t) := \begin{bmatrix}\mathbf{u}(2\ell, t)\\ \mathbf{u}(2\ell+1, t)\end{bmatrix},$$

we get $\mathbf{z}(\ell+1, t+1) = A\mathbf{z}(\ell, t) + B\mathbf{z}(\ell+1, t) + M\tilde{\mathbf{u}}(\ell, t)$,

$$A := \left[\begin{array}{cc|cc} d_{11} & s_{11} & 0 & 0\\ 0 & d_{11} & 0 & 0\\ \hline 0 & s_{21} & d_{22} & s_{22}\\ 0 & 0 & 0 & d_{22}\end{array}\right]$$

$$B := \left[\begin{array}{cc|cc} d_{11} & 0 & 0 & 0\\ s_{11} & d_{11} & 0 & 0\\ \hline 0 & 0 & d_{22} & 0\\ s_{21} & 0 & s_{22} & d_{22}\end{array}\right] \quad M := \left[\begin{array}{cc}\tilde{M} & 0\\ 0 & \tilde{M}\\ \hline 0 & 0\\ 0 & 0\end{array}\right].$$

Finally, by applying the same coordinate transformation $\mathcal{T}$ as in Example 1, and letting $\mathbf{x}(h, k) := \mathbf{z}\big(\mathcal{T}^{-1}(h, k)\big) = \mathbf{z}(h, h + k), \mathbf{u}(h, k) := \tilde{\mathbf{u}}\big(\mathcal{T}^{-1}(h, k)\big) = \tilde{\mathbf{u}}(h, h + k)$, we get the following equation

$$\mathbf{x}(h+1, k+1) = A\mathbf{x}(h, k+1) + B\mathbf{x}(h+1, k) + M\mathbf{u}(h, k+1). \quad (4)$$

2 STRUCTURAL PROPERTIES

Both processes analysed so far have been modelled by means of discrete quarter-plane causal 2D state models which represent particular instances of the class of *2D positive systems*, described by the equation [2]

$$\begin{aligned}\mathbf{x}(h+1, k+1) &= A\mathbf{x}(h, k+1) + B\mathbf{x}(h+1, k),\\ &+ M\mathbf{u}(h, k+1) + N\mathbf{u}(h+1, k)\end{aligned}$$

$h, k \in \mathbb{Z},\ h + k \geq 0$. The *local states* $\mathbf{x}(h, k)$ and the *inputs* $\mathbf{u}(h, k)$ are elements of $\mathbb{R}^n_+$ and $\mathbb{R}^m_+$, respectively, and A, B, M and N are nonnegative matrices. *Initial conditions* are given by assigning a sequence $\mathcal{X}_0 := \{\mathbf{x}(\ell, -\ell) : \ell \in \mathbb{Z}\}$ of nonnegative states on the *separation set* $\mathcal{S}_0 := \{(\ell, -\ell) : \ell \in \mathbb{Z}\}$.

Models (2) and (4), derived in the previous section, share a common feature: the sums of their state transition matrices, $GP + (I - G)P$ and $A + B$ respectively, are (column) substochastic, i.e. the sum of the entries in each column does not exceed one. This property represents the mathematical formalization of the fact that the number of cars as well as the amounts of chemical components cannot increase unless external inputs are applied. More precisely, the i-th component $x_j(h, k)$ of the state $\mathbf{x}(h, k)$ influences only the states in $(h+1, k)$ and $(h, k+1)$, and its contributions, $a_{ij}x_j(h, k)$ in $(h + 1, k)$ and $b_{ij}x_j(h, k)$ in $(h, k + 1)$, $i = 1, \dots, n$, cannot sum up to a quantity greater than $x_j(h, k)$. A complete conservation corresponds to a stochastic matrix sum, whereas leakages motivate the fact that some columns in the matrix sum are not stochastic. This kind of systems represent the two-dimensional analogue of discrete 1D compartmental models, thus motivating the following definition.

Definition 1 A 2D *compartmental system* is a 2D positive system (5) with $A+B$ substochastic.

The above requirement on $A+B$ introduces strong constraints on the spectral properties of the pair (A,B) we aim now to investigate. In the following, it will be convenient to assume that the matrix sum $A+B$ is in Frobenius normal form

$$A+B = \begin{bmatrix} M_{11} & M_{12} & \dots & M_{1r} \\ & M_{22} & & M_{2r} \\ & & \ddots & \vdots \\ & & & M_{rr} \end{bmatrix}, \qquad (5)$$

i.e. is block-triangular, with irreducible blocks M_{ii}. This assumption is not restrictive, as we can reduce $A+B$ to this form by relabelling the state variables.

Proposition 1 *Let $A+B \in \mathbb{R}_+^{n\times n}$ be substochastic, with the block-triangular structure given in (5). Then i) $\rho(M_{ii}) \le 1$ $\forall \imath \in \{1,\dots,r\}$ and $\rho(A+B) \le 1$;*
ii) if $\rho(M_{ii}) = 1$, then M_{ii} is stochastic, $M_{ji} = 0$ $\forall\, j \ne i$, and the maximal modulus eigenvalues of $A+B$ are simple roots of the minimal polynomial of $A+B$.
Proof i) If M is substochastic, there exists a nonnegative matrix Δ s.t. $M+\Delta$ is stochastic, and hence $\rho(M)$, the spectral radius of M, satisfies $\rho(M) \le \rho(M+\Delta) = 1$. Since $A+B$ is substochastic, and this property is inherited by all diagonal blocks M_{ii}, then $\rho(A+B) \le 1$ and $\rho(M_{ii}) \le 1$ for all i.
ii) Assume $\rho(M_{ii}) = 1$, and suppose M_{ii} not stochastic. Then there exists a nonnegative matrix $\Delta \ne 0$ s.t. $M_{ii} + \Delta$ is stochastic, and the irreducibility of M_{ii} guarantees [5] that $\rho(M_{ii}) < \rho(M_{ii}+\Delta) = 1$, a contradiction. As each column of M_{ii} has unitary sum, all entries in the blocks M_{ji}, $j \ne i$, must be zero, and we can assume that $A+B$ has the following structure

$$\left[\begin{array}{ccc|cccc} M_{11} & & & & & & \\ & \ddots & & & * * * & & \\ & & M_{ss} & & & & \\ \hline & & & M_{s+1s+1} & & \dots & M_{s+1r} \\ & \mathbf{0} & & & & & M_{s+2r} \\ & & & & & \ddots & \vdots \\ & & & & & & M_{rr} \end{array}\right]$$
$$= \left[\begin{array}{c|c} A_{11}+B_{11} & A_{12}+B_{12} \\ \hline 0 & A_{22}+B_{22} \end{array}\right] \qquad (6)$$

where the M_{ii}'s, $i = 1,\dots,s$, are irreducible stochastic, while the M_{ii}'s, $i = s+1,\dots,r$, are irreducible substochastic with $\rho(M_{ii}) < 1$.
To prove that every eigenvalue $e^{j\theta}$ of $A+B$ is a simple root of the minimal polynomial, we show that $\ker(e^{j\theta}I - A - B) \equiv \ker(e^{j\theta}I - A - B)^2$. Clearly, as $e^{j\theta}I - A_{22} - B_{22}$ is nonsingular, all vectors in $\ker(e^{j\theta}I - A - B)$ have the entries corresponding to $(e^{j\theta}I - A_{22} - B_{22})$ identically zero.Since all blocks M_{ii}, $i = 1,\dots,s$, are irreducible and stochastic, then $\ker(e^{j\theta}I - M_{ii}) = \ker(e^{j\theta}I - M_{ii})^2$, proving the result. ∎

A 2D compartmental system (5) described by a matrix pair (A,B) whose sum has the structure and the properties of matrix (6) is in *canonical form.* This form suggests some interesting remarks that further motivate the definition of 2D compartmental models.
Consider, first, the 1D compartmental system $\mathbf{z}(t+1) = (A+B)\mathbf{z}(t)$, associated with the matrix sum $A+B$, block partitioned as in (6). Each class of compartments corresponding to some irreducible stochastic block M_{ii}, $i \in \{1,\dots,s\}$, presents no losses, as the total content of the compartments in that class cannot decrease as time goes by. On the other hand, the contents of the remaining compartments decrease to zero, partly due to losses and partly to transfers to lossless compartments. As a consequence, for every initial assignement $\mathbf{z}(0)$, only components corresponding to stochastic blocks can be nonzero in the vector $\mathbf{z}(t)$ as t goes to infinity.

When considering 2D models, it is convenient to think of local states on the same separation set $\mathcal{S}_t := \{(\ell, t-\ell), \ell \in \mathbb{Z}\}$ as representing the contents at time t of compartments $x_1, x_1, \dots, x_n$ at the different space locations ℓ. The content $x_i(\ell, t-\ell)$ of the i-th compartment at time t and location ℓ distributes at time $t+1$, possibly with losses, among the compartments at locations ℓ and $\ell+1$, with rates given by the i-th column of B and A, respectively. By recursively applying this reasoning, we see that $x_i(\ell, t-\ell)$ at time $t+N$ distributes (with losses) among the compartments at locations $\ell, \ell+1, \dots, \ell+N$, and its total contribution to the contents of these compartments is $(A+B)^N \mathbf{e}_i x_i(\ell, t-\ell)$, where $\mathbf{e}_i$ is the i-th canonical vector in $\mathbb{R}^n$. Again, as t goes to infinity, all compartments corresponding to nonstochastic blocks are progressively emptied, whereas those corresponding to stochastic blocks accumulate the whole content (apart from losses) of $x_i(\ell, t-\ell)$.
Similar results hold true, by linearity, when considering all contributions of the states on $\mathcal{S}_t$, thus making clear in what sense the conservation laws hold true when spatial diffusion processes have to be taken into account. As expected, the conservation laws that govern the updating of 2D compartmental models find interesting consequences in terms of stability properties.

Stability definitions for 2D state models refer to the unforced motion determined by an assignement of initial conditions on the separation set $\mathcal{S}_0$. More precisely, we assume that the initial conditions constitute a bounded sequence, which amounts to saying, for 2D positive systems, that all initial local states satisfy $0 \le \mathbf{x}(\ell, -\ell) \le \mathbf{v}$ for some suitable vector $\mathbf{v} \in \mathbb{R}_+^n$. Under this hypothesis, stability concerns the behavior of all local states $\mathbf{x}(h,k)$ as $h+k$ goes to infinity.

Definition 2 A 2D system (5) (a pair (A,B)) is
- *asymptotically stable* if every set $\mathcal{X}_0$ of bounded

initial conditions determines a free evolution which asymptotically estinguishes, i.e. $\mathbf{x}(h,k) \xrightarrow[h+k\to+\infty]{} 0$;

• *(simply) stable* if for every $\varepsilon > 0$ there is $\delta > 0$ s.t. any sequence of initial conditions satisfying $\mathbf{x}(\ell,-\ell) < \delta\mathbf{u}_n$, with $\mathbf{u}_n := [1\ \dots\ 1]^T$, determines a free evolution for which $\mathbf{x}(h,k) < \varepsilon\mathbf{u}_n,\ h+k \geq 0$.

The characterization of asymptotic stability given in Proposition 2 below has been presented in [6]. Points ii) and iii) include additional facts, needed for a complete analysis of 2D compartmental systems stability.

Proposition 2 *Consider a 2D positive system (5), with state transition matrices $A, B \in \mathbb{R}^{n\times n}$.*
i) (A,B) is asymptotically stable iff $\rho(A+B) < 1$;
ii) if (A,B) is stable, then $\rho(A+B) \leq 1$;
iii) if $\rho(A+B) \leq 1$ and in (5) condition $\rho(M_{ii}) = 1$ implies $M_{ji} = 0,\ j = 1,\dots,i-1$, then (A,B) is stable.

The proof depends upon the following easy lemmas.

Lemma 1 *Suppose that in (5) all initial conditions satisfy $\mathbf{x}(\ell,-\ell) \leq \mathbf{v}$, for some $\mathbf{v} \in \mathbb{R}^n_+$; then all states of the corresponding free evolution satisfy $\mathbf{x}(h,k) \leq (A+B)^{h+k}\mathbf{v}$.* ∎

Lemma 2 *If M is an $n\times n$ nonnegative matrix, for every $\lambda > \rho(M)$ there is an $n\times n$ positive matrix L s.t. $M^i < \lambda^i L, \forall\ i \in \mathbb{N}$. Furthermore, if M has a strictly positive eigenvector, in particular is irreducible, the above property holds also for $\lambda = \rho(M)$.* ∎

Proof of Proposition 2 i) For every bounded sequence $\mathcal{X}_0$ there exists $c > 0$ s.t. $\mathbf{x}(\ell,-\ell) \leq c\,\mathbf{u}_n, \forall\ \ell \in \mathbb{Z}$. Moreover, if $\rho(A+B) < 1$, for every λ satisfying $\rho(A+B) < \lambda < 1$, there exists, by Lemma 2, a positive matrix L s.t. $(A+B)^i < \lambda^i L, \forall\ i \in \mathbb{N}$. So, from Lemma 1, we get $\mathbf{x}(h,k) \leq c\ \lambda^{h+k} L\mathbf{u}_n \xrightarrow[h+k\to+\infty]{} 0$.
Vice versa, assume $\rho(A+B) \geq 1$ and consider in (5) a block M_{ii} with $\rho(M_{ii}) = \rho(A+B)$. Let $\mathbf{v}_i \gg 0$ denote a Perron-Frobenius eigenvector of M_{ii} and $\mathbf{v}$ the n-dimensional vector, partitioned conformably with $A+B$, whose i-th block coincides with $\mathbf{v}_i$, while the remaining ones are zero. The free state evolution corresponding to $\mathcal{X}_0 = \{\mathbf{x}(\ell,-\ell) = \mathbf{v}, \ell \in \mathbb{Z}\}$, does not converge to zero.
ii) If $\rho(M_{ii}) > 1$, the vector $M_{ii}^j(\delta\mathbf{v}_i)$ diverges, no matter how small $\delta > 0$ is chosen. Hence the sequence $\mathbf{x}(\ell,-\ell) = \delta\mathbf{v},\ \forall\ \ell \in \mathbb{Z}$, with $\mathbf{v}$ defined as in i), produces diverging components in $\mathbf{x}(h,k)$ as $h+k$ goes to infinity, which prevents (A,B) from being stable.
iii) Suppose that $A+B$ is in the form (6) and let n_1 be the size of A_{11} and B_{11}. As $A_{11}+B_{11}$ admits a strictly positive eigenvector $\mathbf{w}$ which corresponds to the eigenvalue $\rho(A_{11}+B_{11}) = 1$, Lemma 2 gives $(A_{11}+B_{11})^j \leq [\rho(A_{11}+B_{11})]^j L_{11} = L_{11},\ \forall j \in \mathbb{N}$, with L_{11} a suitable $n_1 \times n_1$ positive matrix. Again, by Lemma 2, there exist $\lambda \in (0,1)$ and an $(n-n_1)\times(n-n_1)$ positive matrix L_{22} s.t. $(A_{22}+B_{22})^j < \lambda^j L_{22}, \forall j \in \mathbb{N}$. Upon assuming (which is not a restriction) $\mathbf{w} \geq \mathbf{u}_{n_1}$, we get $(A+B)^j\mathbf{u}_n \leq (A+B)^j \begin{bmatrix} \mathbf{w} \\ \mathbf{u}_{n-n_1} \end{bmatrix} \leq \begin{bmatrix} \mathbf{w} + \frac{1}{1-\lambda}L_{11}(A_{12}+B_{12})\mathbf{u}_{n-n_1} \\ L_{22}\mathbf{u}_{n-n_1} \end{bmatrix}$,which implies the existence of $b > 0$ s.t. $(A+B)^j\mathbf{u}_n \leq b\ \mathbf{u}_n, \forall j \in \mathbb{N}$. Assuming that all states in $\mathcal{X}_0$ satisfy $\mathbf{x}(\ell,-\ell) \leq \mathbf{u}_n$, Lemma 1 and the last inequality imply $\mathbf{x}(h,k) \leq b\mathbf{u}_n, h+k \geq 0$. So, given $\varepsilon > 0$, simple stability condition is fulfilled by selecting $\delta = \varepsilon/b$. ∎

Corollary *A 2D compartmental system with state transition matrices A and B is always stable. It is asymptotically stable iff $\rho(A+B) < 1$.*
Proof Since $A+B$ is substochastic, $\rho(A+B) \leq 1$. Moreover, $A+B$ is cogredient to form (6). So, both conditions of point iii) in Proposition 2 are met. The second statement has already been proved. ∎

3 CONCLUSIONS

This paper makes a first attempt to introduce 2D system methods in the analysis of distributed processes that exibit compartmental structure. A couple of examples has been considered, enlightening concrete applications of the rich body of 2D theory in this area. A distinguishing feature, with respect to procedures based the discretization of ODEs or PDEs models, is that a first principle derivation of the discrete model is obtained, based on balance equations among different compartments.
The theoretical results here presented are still far from complete, and further investigations should take into account state reconstruction and feedback control. New advances will hopefully lead to satisfactory algorithms for monitoring and control purposes.

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