SOME ASPECTS OF THE ALGEBRAIC STRUCTURE OF 2D CONVOLUTIONAL CODES

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1. Abstract

In recent times the 2D system analysis has constituted a stimulating research topic, both for the theoretical interest and the applications in two-dimensional digital signal processing ([1], [2]).

As well known, 2D discrete filtering usually resorts to input/output ARMA representations or to transfer matrices whose entries are rational functions in two variables. In the last few years an innovative approach to the 2D system representation has been proposed by J.C.Willems and P.Rocha ([3], [4]). They have introduced a mathematical framework for analyzing the trajectories of dynamical systems on the discrete plane and for representing their evolution laws, that seems suitable for studying arbitrary two-dimensional data structures.

The convolutional codes theory has largely made use of polynomial matrix algebra and some other tools borrowed from Linear Systems Analysis, thus supplying further results whose interest largely exceeds the original field of research. Quite recently, G.D.Forney ([5]) has applied Willems viewpoint to the investigation of the properties of convolutional codes over groups, specified through the set of all their possible trajectories (i.e. the behaviour).

On the same streamline, this paper aims to give a preliminary report on 2D convolutional codes, by providing some new results and pointing out open problems and directions of further investigation.

The first part of the paper is devoted to the study of some properties a good encoder must be endowed with. A natural requirement is the external controllability (see Willems, [5]), that is the possibility of generating code sequences whose restriction to a finite region of the plane entails no loss of freedom on the values assumed far away from that region. A controllable code C can be represented as the image of a 2D polynomial matrix

$$C = \mathrm{Im} G(\sigma_1, \sigma_2)$$

where σ_1 and σ_2 are the shift operators along the coordinate axes.

A further requirement for a good encoder is to produce code sequences which, if received incorrupted, allow an exact reconstruction of the generating information sequence. If we look at the polynomial encoder as a linear input/output map on 2D sequence spaces, the above property entails the injectivity of the map. It can be

shown that this is equivalent to the zero primeness of the matrix $G(z_1, z_2)$, i.e. to the fact that the variety of the maximal order minors of $G(z_1, z_2)$ is empty.

Given any 2D code C, there exist many zero prime polynomial matrices which can be equivalently used for its representation. Any two such matrices differ from each other in a unimodular 2D left factor.

A natural goal is to extract out of the set of all the equivalent matrices for C, those endowed with properties relevant either for the structure of the input/output map or for the complexity of the circuital synthesis. Interestingly enough, the equivalence among properties like minimality, predictable degree and row properness of the leading terms matrix in the 1D, no longer holds in the 2D case.

It is well known that an encoder $k \times n$ is systematic if it has I_k as a submatrix: given a polynomial encoder $G(z_1, z_2)$, we will provide a criterion for testing whether it is equivalent to a systematic one. Moreover, when these conditions are not fulfilled, we investigate the possibility of modifying the encoder by adding to it a minimal number of columns, so as to make it equivalent to a systematic one.

The second part of the paper is devoted to the problem of obtaining for the encoder $G(z_1, z_2)$ a state space realization of the following kind:

$$x(h+1, k+1) = A_1 x(h, k+1) + A_2 x(h+1, k) + B_1 u(h, k+1) + B_2 u(h+1, k) y(h, k) = Cx(h, k)$$

i.e. a sextuple of matrices $(A_1, A_2, B_1, B_2, C, E)$ such that

$$G(z_1, z_2) = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + E$$

Solving this problem constitutes a fixed course towards the circuital synthesis of the 2D encoder: it's clear that we are interested in realizations of minimal dimension and among them in finite memory realizations. Finite memory realizations, namely those satisfying the condition $\det(I - A_1 z_1 - A_2 z_2) = 1$, play an important role, since they reach the zero state in a finite number of steps after the zeroing of the input signal.

Finally the concepts of 2D dual code and encoder are introduced: the space of the 2D sequences generated by the dual encoder is the orthogonal to the sequence space C. While the dual code is uniquely determined by the encoder $G(z_1, z_2)$, this no longer holds for the dual encoder. By applying the Quillen-Suslin Theorem ([6]), we can row border the zero prime matrix $G(z_1, z_2)$ so as to obtain a 2D unimodular matrix: its polynomial inverse provides both the dual encoder $D(z_1, z_2)$ and the right inverse of $G(z_1, z_2)$. C can be expressed not only as the image of G, but also as the kernel of $D^T(z_1, z_2)$.

2. References

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