On rational functions that admit a positive realization

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Abstract

The paper investigates the possibility of synthesizing a positive system in state space form as a (series, parallel and feedback) interconnection of a finite number of elementary positive systems.

1 Introduction

A positive system is a system in which input, state and output variables are positive (or at least nonnegative) in value. The positivity constraint arises naturally when modelling real systems, whose variables represent quantities that are intrinsically nonnegative, such as pressures, concentrations, population levels, etc..

An interesting issue, that received large attention in the last few years, is the so called "positive realization problem". It consists in determining a set of necessary and sufficient conditions on the input/output behavior of a linear system, guaranteeing the existence of a positive state space realization. It turns out that, when one assumes that all Markov coefficients of a transfer function $w(z)$ are nonnegative, the existence of a positive realization depends on the polar structure of $w(z)$. As a matter of fact, the coefficients assumption, i.e. that the power series expansion of $w(z)$ has to be an element of the semiring $\mathbb{R}_+[[z^{-1}]]$, appears somehow inhomogeneous w.r.t. the remaining condition.

In a different context, when one analyzes the possibility of realizing a transfer function by means of a passive electrical network, it is well known that $w(z)$ has to be positive real - a condition that only involves the structure of $w(z)$ as an holomorphic function, and constructive techniques have been devised for synthesizing an electrical network, based on the iterative removal of basic circuit elements (i.e. resistors, inductors, capacities and ideal transformers) or suitable combinations thereof , in such a way that the complexity of the remaining transfer function is progressively reduced.
In this paper we show that a transfer function that admits a positive realization can be synthesized as a finite interconnection of elementary first order positive systems, a result that suggests the possibility of arriving at a positive synthesis procedure based on the sequential removal of elementary positive systems.

2 Positive realizations and elementary positive systems interconnections.

Let $\mathbb{R}_p(z)$ be the ring of proper rational functions and, for all $p(z)$ and $c(z)$ in $\mathbb{R}_p(z)$ such that $1 \neq p(\infty)c(\infty)$, introduce the binary operation

$$p(z)\diamondsuit c(z) = \frac{p(z)}{1 - p(z)c(z)}$$

It is clear that $p(z)\diamondsuit c(z)$ is again a proper rational function, i.e. the transfer function of the (positive) feedback connection of Fig.1.

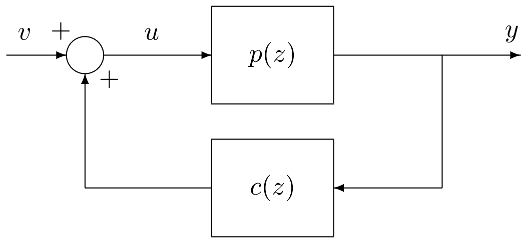

Fig. 1

Denote by $\mathcal{P}$ the smallest semiring of $\mathbb{R}_p(z)$ with the following properties:

1. $\mathcal{P}$ is closed w.r.t. $\diamondsuit$ operation for all pairs $(p(z), c(z))$ such that
$$p(\infty)c(\infty) < 1$$
2. $\mathcal{P}$ includes $\mathbb{R}_+$
3. $\mathcal{P}$ includes z (or, equivalently, all functions $\frac{\alpha}{z-\beta}$ with α and β nonnegative).

Proposition 1 *The set $\mathcal{R}$ of proper rational functions that admit a discrete time positive realization is a semiring that includes $\mathcal{S}$.*

PROOF Both the sum and the product of two rational functions in $\mathcal{R}$ are positively realizable, via the parallel and the series connection of a couple of positive systems, respectively. Hence $\mathcal{R}$ is a subsemiring of $\mathbb{R}_p(z)$. Clearly $\mathcal{R}$ includes $\mathbb{R}_+$, and $\frac{\alpha}{z-\beta}$ if α and β are nonnegative.
Moreover, if $\Sigma_1 = (F_1, G_1, H_1, j_1)$ is a positive realization of $p(z)$, $\Sigma_2 = (F_2, G_2, H_2, j_2)$ is a positive realization of $c(z)$ and $j_1 j_2 < 1$, the system $\Sigma = (F, G, H, j)$ with

$$
\begin{aligned}
F &= \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} \frac{1}{1-j_1 j_2} & \frac{j_2}{1-j_1 j_2} \\ \frac{J_1}{1-j_1 j_2} & \frac{1}{1-j_1 j_2} \end{bmatrix} \begin{bmatrix} 0 & H_2 \\ H_1 & 0 \end{bmatrix}, \\
G &= \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} \frac{1}{1-j_1 j_2} \\ \frac{j_1}{1-j_1 j_2} \end{bmatrix} \\
H &= [\, H_1 \quad 0 \,] + [\, j_1 \quad 0 \,] \begin{bmatrix} \frac{1}{1-j_1 j_2} & \frac{J_2}{1-j_1 j_2} \\ \frac{j_1}{1-j_1 j_2} & \frac{1}{1-j_1 j_2} \end{bmatrix} \begin{bmatrix} 0 & H_2 \\ H_1 & 0 \end{bmatrix} \\
j &= \frac{j_1}{1 - j_1 j_2}
\end{aligned}
$$

provides a positive realization of $p(z) \diamondsuit c(z) = \frac{p(z)}{1-p(z)c(z)}$.

Suppose now that $\Sigma = (F, G, H, j)$ is a single input/single output positive system of dimension $n \geq 2$ and assume first that F is block triangular

$$F = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix},$$

where F_{11} and F_{22} have dimensions $n_1 \times n_1$ and $n_2 \times n_2$ respectively. Upon partitioning G and H in a conformable way

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = [\, H_1 \quad H_2 \,]$$

we see that the transfer function of Σ can be written as

$$
\begin{aligned}
w(z) &= [\, H_1 \quad H_2 \,] \\
&\times \begin{bmatrix} zI - F_{11}^{-1} & 0 \\ (zI - F_{22})^{-1} F_{21} (zI - F_{11})^{-1} & (zI - F_{22})^{-1} \end{bmatrix} \\
&\times \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} + j \\
&= \Big[H_1 (zI - F_{11})^{-1} G_1 \Big] + \big[H_2 (zI - F_{22})^{-1} G_2 + j \big] \\
&+ \Big[H_2 (zI - F_{22})^{-1} I \Big] \Big[F_{21} (zI - F_{11})^{-1} G_1 \Big] \\
&= w_1(z) + w_2(z) + w_{12}(z).
\end{aligned}
$$

Here $w_{12}(z) = H_2(zI - F_{22})^{-1} F_{21} (zI - F_{11})^{-1} G_1$ can be viewed as the transfer matrix of the series of an n_1-dimensional single-input/n_2-outputs positive system $\Sigma' = (F_{11}, G_1, F_{21})$ feeding an n_2-dimensional n_2-inputs/single-output positive system $\Sigma'' = (F_{22}, I_{n_2}, H_2)$.

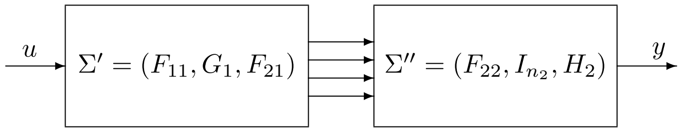

Fig. 2

The above series of two positive multivariable systems can be replaced by the parallel of n_2 series of scalar positive systems: if $F_{21}^{(i)}$ and $\mathbf{e}_i$ denote the i-th row and the i-th column of F_{21} and I_{n_2} respectively, $W_{21}(z)$ can be written as $\sum_{i=1}^{n_2} H_2(zI - F_{22})^{-1} \mathbf{e}_i F_{21}^{(i)} (zI - F_{11})^{-1} G_1$ and the i-th term in the sum is realized by the series of $\Sigma_{1i} = (F_{11}, G_1, F_{21}^{(i)})$ feeding $\Sigma_{2i} = (F_{22}, \mathbf{e}_i, H_2)$.

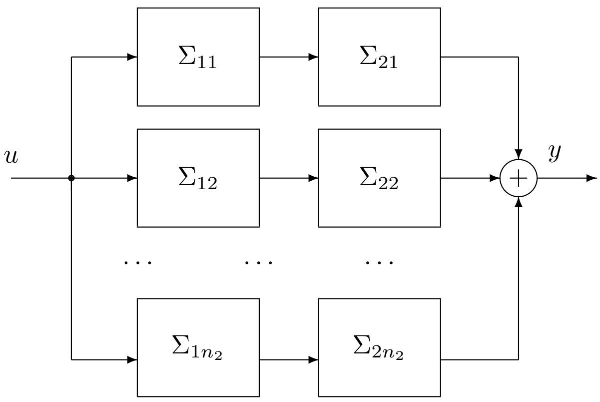

Fig. 3

Basing on these considerations, we conclude that the realization of $w(z) = w_1(z) + w_2(z) + w_{21}(z)$ reduces to a number of series and parallel interconnections of suitable positive scalar systems with dimension lower than n. In addition, if the above procedure can be iteratively applied to the lower dimensional systems we obtain, we end up with a system consisting of series and parallels of one-dimensional positive systems. In general, however, this is not the case, and we have to consider positive realizations in which th F matrix is not block triangular.

If so, in the positive system $\Sigma = (F, G, H, j)$ we partition F into

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & f_{22} \end{bmatrix},$$

where f_{22} is the element in position (n, n). As a consequence, F_{12} and F_{21} are a column and a row matrix, respectively and G and H are conformably rewritten as

$$G = \begin{bmatrix} G_1 \\ g_2 \end{bmatrix}, \quad H = [\, H_1 \quad h_2 \,]$$

The transfer function of Σ can be rewritten as

$$\begin{aligned} w(z) &= [\,H_1 \quad h_2\,] \begin{bmatrix} zI - F_{11} & -F_{12} \\ -F_{21} & z - f_{22} \end{bmatrix}^{-1} + j \\ &= [\,H_1 \quad h_2\,] \begin{bmatrix} M_{11}(z) & M_{12}(z) \\ M_{21}(z) & m_{22}(z) \end{bmatrix} \begin{bmatrix} G_1 \\ g_2 \end{bmatrix} + j \end{aligned}$$

with

$$\begin{aligned} M_{11}(z) &= (zI - F_{11})^{-1} \\ &+ (zI - F_{11})^{-1} F_{12} X(z)^{-1} F_{21} (zI - F_{11})^{-1} \\ M_{12}(z) &= (zI - F_{11})^{-1} F_{12} X(z)^{-1} \\ M_{21}(z) &= X(z)^{-1} F_{21} (zI - F_{11})^{-1} \\ m_{22}(z) &= X(z)^{-1} \end{aligned}$$

and

$$X(z) := (z - f_{22}) - F_{21}(zI - F_{11})^{-1} F_{12}$$

denotes the Schur complement of $(zI - F)$.
We therefore have

$$\begin{aligned} w(z) &= j + h_2 X(z)^{-1} g_2 \\ &+ H_1 (zI - F_{11})^{-1} G_1 \\ &+ H_1 (zI - F_{11})^{-1} F_{12} X(z)^{-1} F_{21} (zI - F_{11})^{-1} G_1 \\ &+ h_2 X(z)^{-1} F_{21} (zI - F_{11})^{-1} G_1 \\ &+ H_1 (zI - F_{11})^{-1} F_{12} X(z)^{-1} g_2 \end{aligned}$$

The inverse of the Schur complement is given by

$$\begin{aligned} X(z)^{-1} &= \frac{1}{(z - f_{22}) - F_{21}(zI - F_{11})^{-1} F_{12}} \\ &= \frac{\dfrac{1}{z - f_{22}}}{1 - \dfrac{1}{z - f_{22}} F_{21}(zI - F_{11})^{-1} F_{12}} \\ &= \frac{1}{z - f_{22}} \diamondsuit \Big[F_{21}(zI - F_{11})^{-1} F_{12} \Big] \end{aligned}$$

and hence can be realized by the feedback interconnection of two positive systems of dimension 1 and $n-1$ respectively. It is now clear that each term in the above expression for $w(z)$ is the transfer function of a series and/or feedback interconnection of a finite number of positive systems with dimension not greater than $n-1$, and hence $w(z)$ can be synthesized as the parallel of such systems.

The iteration of the above procedure provides a proof of the following

Proposition 2 *Every transfer function in $\mathcal{R}$ is a (series, parallel and feedback) interconnection of a finite number of systems whose transfer functions are either elements of $\mathbb{R}_+$ or rational functions $\alpha/(z-\beta)$, with α and β in $\mathbb{R}_+$.*

3 References

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