Uncertainty in Bipolar Preference Problems

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Abstract. Preferences and uncertainty are common in many real-life problems. In this paper, we focus on bipolar preferences and on uncertainty modelled via uncontrollable variables, and we assume that uncontrollable variables are specified by possibility distributions over their domains. To tackle such problems, we concentrate on uncertain bipolar problems with totally ordered preferences, and we eliminate the uncertain part of the problem, while making sure that some desirable properties hold about the robustness of the problem and its relationship with the preference of the optimal solutions. We also consider several semantics to order the solutions according to different attitudes with respect to the notions of preference and robustness.

1 Introduction

Real-life problems present several kinds of preferences and may be affected by uncertainty. In this paper, we focus on problems with positive and negative preferences with uncertainty.

Bipolar preferences [23, 18, 1–3, 13, 14, 7, 8, 10] and uncertainty [19, 15, 28, 20] appear in many application fields, such as satellite scheduling, logistics, and production planning. Moreover, in multi-agent problems, agents may express their preferences in a bipolar way, and variables may be under the control of different agents. To give a specific example, just consider a conference reviewing system, where usually preferences are expressed in a bipolar scale. Uncertainty can arise for the number of available conference rooms at the time of the acceptance decision. The goal could be to select the best papers while ensuring that they all can be presented.

Bipolarity is an important topic in several domains, e.g., psychology [27, 24, 11], multi-criteria decision making [23], and more recently also in AI (in areas such as argumentation [18, 1] and qualitative reasoning [2, 3, 13, 14]). Preferences on a set of possible choices are often expressed in two forms: positive and negative statements. In fact, in many real-life situations agents express what they like and what they dislike, thus often preferences are bipolar.

In this paper, to handle bipolarity, we use the formalism presented in [7, 8, 10]. Related but different formalisms to achieve a similar goal can be found in [23, 1–3, 13, 14]. The considered formalism generalizes to positive and negative preferences the soft constraints formalism [6], which is able to model problems with one kind of preferences (i.e., negative preferences). Thus, each partial instantiation within a constraint will be associated to either a positive or a negative preference. For example, when buying a house, we may like very much to live in the country, but we may also not like to have to take a bus to go to work, and be indifferent to the color of the house. Thus we will give a preference level (either positive, or negative, or indifference) to each feature of the house, and then we will look for a house that has the best combined preference overall.

Another important feature, which arises in many real world problems, is uncertainty. In [23, 1–3, 13, 14] the authors handle bipolarity but not the presence of uncertainty. In this paper, we consider both bipolarity and uncertainty. We model uncertainty by the presence of *uncontrollable* variables. This means that the value of such variables will not be decided by us, but by Nature or by some other agent. Thus a solution of such problems will not be an assignment to all the variables but only to the controllable ones. A typical example of an uncontrollable variable, in the context of satellite scheduling or weather prediction, is a variable representing the time when clouds will disappear. A more general setting in which uncertainty occurs are scheduling problems, which constrain the order of execution of various activities, and where the durations of some activities may be uncertain [15]. In this case the goal is to define a schedule which is the most robust with respect to the uncertainty.

Although we cannot choose the value for such uncontrollable variables, usually we have some information on the plausibility of the values in their domains. In [19] the information over uncontrollable variables, which is not bipolar, is given in terms of probability distributions. In this paper, we model this information by a possibility distribution over the values in the domains of such variables. Possibilities are useful when probability distributions are not available, and provide upper and lower bounds to probabilities [28].

In this paper we focus on problems with this kind of uncertainty, and that contain positive and negative preferences. We call them *uncertain bipolar problems*. To tackle such problems, we generalize to bipolar preferences the approach to handle fuzzy preferences (that are a special kind of negative preferences) and uncertainty presented in [26, 25]. In particular, we generalize to the bipolar context the notions of preference and robustness for the solutions, as well as properties that such notions should respect in relation to the solution ordering, and the procedure used to compute preference and robustness degrees. First, we generalize the approach presented in [26] to uncertain bipolar problems where the set of the positive preferences and the set of the negative preferences are closed intervals of $\mathbb{R}$. Then, we use abstraction techniques and Galois connection properties [5] to generalize the procedure also to uncertain bipolar problems where the set of positive/negative preferences are generic totally ordered sets.

Our approach follows the one presented in [26]. More precisely, given an uncertain bipolar problem, the uncontrollable part of the problem is removed and new constraints on the controllable part are added. Thus, we obtain a bipolar problem without uncertainty and with additional constraints. Such additional constraints are considered to define the robustness of the problem. Starting from this problem, we define the preference and the robustness of the solutions of the initial uncertain problem, and we show that they satisfy some desired properties. Moreover, we consider some semantics that use such notions to order the solutions, and we show that they satisfy desired properties on the solution ordering. In particular, they allow us to distinguish between highly preferred solutions which are not robust, and robust but not preferred solutions. Also, they guarantee that, if there are two solutions s and s' with the same robustness (resp., the same preference), and the preference (resp., the robustness) of s is better than the preference (resp., the robustness) of s', then s is considered better than s'.

The paper is structured as follows. Section 2 provides the readers with the main notions about positive, negative, and bipolar properties, bipolar preference problems, soft constraint problems with uncertainty and their properties, as well as the approach of [26] for removing uncertainty in uncertain fuzzy CSPs. Then, Section 3 introduces the notion of uncertain bipolar problems, while Section 4 defines some desirable properties of such problems. Section 5 describes the approach to solve uncertain bipolar problems, while Section 6 defines the notions of preference and robustness of such problems, and relates them to the properties proposed in Section 4. Section 7 studies some possible semantics for uncertain bipolar problems. Then, Section 8 extends the overall approach to more general bipolar preference structures, and Section 9 summarizes the main results and gives some hints for possible lines of future work.

This paper is a revised and extended version of [9]. In particular, while [9] shows only a procedure for handling bipolar preference problems where the sets of positive and negative preferences are two closed intervals of $\mathbb{R}$, this paper proposes also a procedure to handle bipolar problems where the set of positive/negative preferences are generic totally ordered structures, by using abstraction techniques and galois connections properties [5].

2 Background

We now give some basic notions on bipolar preference problems [7, 8, 10] and on uncertain soft (fuzzy) problems [26, 25].

2.1 Negative preferences

Bipolar preference problems [7, 8, 10] are based on a bipolar preference structure, which allows to handle both positive and negative preferences. This structure contains two substructures, one for each kind of preferences.

When dealing with negative preferences, two main properties should hold: combination should bring to worse preferences, and indifference should be better than all the other negative preferences. These properties can be found in a c-semiring [6], which is the structure used to represent soft constraints.

A *c-semiring* is a tuple $(A, +, \times, \mathbf{0}, \mathbf{1})$ where: A is a set and $\mathbf{0}, \mathbf{1} \in A$; $+$ is commutative, associative, idempotent, $\mathbf{0}$ is its unit element, and $\mathbf{1}$ is its absorbing element; $\times$ is associative, commutative, distributes over $+$, $\mathbf{1}$ is its unit element and $\mathbf{0}$ is its absorbing element. Consider the relation $\leq_S$ over A such that $a \leq_S b$ iff $a + b = b$. Then: $\leq_S$ is a partial order; $+$ and $\times$ are monotonic on $\leq_S$; $\mathbf{0}$ is its minimum and $\mathbf{1}$ its maximum. Informally, the relation $\leq_S$ gives us a way to compare (some of the) tuples of values and constraints. In fact, when $a \leq_S b$, we will say that *b is better than a*.

Given a c-semiring $S = (A, +, \times, \mathbf{0}, \mathbf{1})$, a finite set D (the domain of the variables), and an ordered set of variables V, a soft constraint is a pair $\langle def, con\rangle$ where $con \subseteq V$ and $def : D^{|con|} \to A$. Therefore, a soft constraint specifies a set of variables (the ones in con), and assigns to each tuple of values of D of these variables an element of A. A *soft constraint satisfaction problem* (SCSP), denoted by $\langle S, V, C\rangle$, is a set of soft constraints C based on the c-semiring S, which is defined over a set of variables V. For example, fuzzy CSPs [21] are SCSPs that can be modeled by choosing the c-semiring $S_{FCSP} = ([0, 1], max, min, 0, 1)$.

In a c-semiring there is an element which combined with every other preference returns such a preference, i.e., there is an element that acts as indifference. Such an element is $\mathbf{1}$. In fact, $\forall a \in A$, $a\times\mathbf{1} = a$. Moreover, in a c-semiring holds a desired property for negative preferences, that is, the combination between preferences is worse than the considered preferences (in fact, $\forall a, b \in A$, $a \times b \leq a, b$). This interpretation is very natural when considering, for example, the weighted c-semiring $(R^+, min, +, +\infty, 0)$, where preferences are real positive numbers interpreted as costs. Such costs are combined via the sum ($+$) and the best costs are the lower ones (min). In this case preferences are costs and thus negative preferences, and the sum of the cost costs is worse in general than these costs, since we want to minimize the sum of the cost.

The interpretation above is also natural when considering, the fuzzy c-semiring $([0, 1], max, min, 0, 1)$, where preferences are in $[0, 1]$, are combined via the minimum operator and the best preferences are the higher ones (max). In fact, in this case the combination of preferences is worse in general than these preferences, since it is equal to the worst one of these preferences w.r.t. the ordering induced by the additive operator (that is, max) of the c-semiring.

From now on, a standard c-semiring will be used to model negative preferences, denoted as: $(N, +_n, \times_n, \bot_n, \top_n)$.

2.2 Positive preferences

When dealing with positive preferences, two main properties should hold: combination should bring to better preferences, and indifference should be lower than all the other positive preferences.

These properties can be found in a *positive preference structure* [7, 8, 10], which is a tuple (P, $+_p$, $\times_p$, $\bot_p$, $\top_p$) s.t. P is a set and $\top_p$, $\bot_p \in P$; $+_p$, the additive operator, is commutative, associative, idempotent, with $\bot_p$ as its unit element ($\forall a \in P$, $a +_p \bot_p = a$) and $\top_p$ as its absorbing element ($\forall a \in P$, $a +_p \top_p = \top_p$); $\times_p$, the multiplicative operator, is associative, commutative and distributes over $+_p$ ($a \times_p (b +_p c) = (a \times_p b) +_p (a \times_p c)$), with $\bot_p$ as its unit element and $\top_p$ as its absorbing element[1].

[1] The absorbing nature of $\top_p$ can be derived from the other properties.

The additive operator of this structure has the same properties as the corresponding one in c-semirings, and thus it induces a partial order over P in the usual way: $a \leq_p b$ iff $a +_p b = b$. This allows to prove that $+_p$ is monotonic ($\forall a, b, d \in P$ s.t. $a \leq_p b$, $a \times_p d \leq_p b \times_p d$) and that $+_p$ is the least upper bound in the lattice $(P, \leq_p)$ ($\forall a, b \in P$, $a \times_p b \geq_p a +_p b \geq_p a, b$).

On the other hand, $\times_p$ has different properties w.r.t. $\times_n$: its absorbing element is now the best element in the ordering $(\top_p)$, while its unit element, that can model indifference, is the worst element $(\perp_p)$. These are exactly the desired properties for combination and indifference of positive preferences. An example of a positive preference structure is ($\Re^+$,max,sum,0,$+\infty$), where preferences are positive real numbers aggregated with sum and compared with max (i.e., the best preferences are the highest ones). Another example is ($[0, 1]$,max,max,0,1), where preferences are positive real numbers aggregated and compared with max.

2.3 Bipolar preferences

When we deal with both positive and negative preferences, the same properties described above for a single kind of preferences should continue to hold. Moreover, all the positive preferences should be better than all the negative ones and there should exist an operator which allows for the compensation between positive and negative preferences. These properties can be obtained by considering the bipolar preference structure presented below, that links the previous two structures by setting the highest negative preference to coincide with the lowest positive preference to model indifference.

A *bipolar preference structure* [7, 8, 10] is a tuple $(N, P, +, \times, \perp, \Box, \top)$ where, $(P, +_{|_P}, \times_{|_P}, \Box, \top)$ is a positive preference structure; $(N, +_{|_N}, \times_{|_N}, \perp, \Box)$ is a c-semiring; $+ : (N \cup P)^2 \longrightarrow (N \cup P)$ is an operator s.t. $a_n + a_p = a_p$, $\forall a_n \in N$ and $a_p \in P$; it induces a partial ordering on $N \cup P$: $\forall a, b \in P \cup N$, $a \leq b$ iff $a + b = b$; $\times : (N \cup P)^2 \longrightarrow (N \cup P)$ (called the *compensation operator*) is a commutative and monotonic ($\forall a, b, c \in N \cup P$, if $a \leq b$, then $a \times c \leq b \times c$) operator.

In the following, we will write $+_n$ instead of $+_{|_N}$ and $+_p$ instead of $+_{|_P}$. Similarly for $\times_n$ and $\times_p$. When $\times$ is applied to a pair in $(N \times P)$, we will sometimes write $\times_{np}$.

Note that the compensation operator may not be associative. This is due to the fact that one wants to leave complete freedom to choose the positive and negative algebraic structures. However, in some situations associativity could be desirable. In such a case one can build a bipolar structure with associative compensation operator, by following the procedure presented in [8, 10].

From the monotonicity of the compensation operator it follows that the combination of a positive and a negative preference is a preference which is higher than, or equal to, the negative one and lower than, or equal to, the positive one.

An example of bipolar structure is the tuple (N=$[-1, 0]$, P=$[0, 1]$, $+$=max, $\times$, $\perp$=-1, $\Box$=0, $\top$=1), where $\times$ is such that $\times_p$= max, $\times_n$=min and $\times_{np}$=sum. Negative preferences are between -1 and 0, positive preferences between 0 and 1, compensation is sum, and the order is given by max. In this case $\times$ is not associative.

Note that, when the preferences are totally ordered, operators $\times_n$ and $\times_p$ described here correspond resp. to the *t-norm* and *t-conorm* considered in [22], and requiring that the compensation operator is associative, then it corresponds to the $uninorm$ operator considered in [22].

2.4 Bipolar preference problems

A bipolar constraint is a constraint where each assignment of values to its variables is associated to one of the elements in a bipolar preference structure. Given a bipolar preference structure $S = (N, P, +, \times, \perp, \square, \top)$, a finite set D (the domain of the variables), and an ordered set of variables V, a *bipolar constraint* is a pair $\langle def, con \rangle$ where $con \subseteq V$ and $def : D^{|con|} \rightarrow (N \cup P)$. A *bipolar CSP* (BCSP) $\langle S, V, C \rangle$ is a set of variables V and a set of bipolar constraints C over V defined on the bipolar structure S.

An RBCSP $\langle S, V, C_1, C_2 \rangle$ is a BCSP over the bipolar structure S, where the set of variables is V and the set of bipolar constraints is $C_1 \cup C_2$.

Given a subset of variables $I \subseteq V$, and a bipolar constraint $c = \langle def, con \rangle$, the *projection of* c over I, written $c \Downarrow_I$, is a new bipolar constraint $\langle def', con' \rangle$, where $con' = con \cap I$ and $def(t') = \sum_{\{t | t \downarrow_{con'} = t'\}} def(t)$. In particular, the scope, con', of the projection constraint contains the variables that con and I have in common, and thus $con' \subseteq con$. Moreover, the preference associated to each assignment to the variables in con', denoted with t', is the best one among the preferences associated by def to any completion of t', t, to an assignment to con. The notation $t \downarrow_{con'}$ indicates the subtuple of t on the variables of con'. For example, if $con = X \cup Y$, $con' = X$, and $t = (X = a, Y = b)$, then $t \downarrow_X = a$.

A *solution* of a BCSP $\langle S, V, C \rangle$ is a complete assignment to all variables in V, say s. Its overall preference is $ovpref(s) = ovpref_p(s) \times ovpref_n(s) = (p_1 \times_p \ldots \times_p p_k) \times (n_1 \times_n \ldots \times_n n_l)$, where, for $i := 1, \ldots, k$, $p_i \in P$, for $j := 1, \ldots, l$, $n_j \in N$, and $\exists \langle def_i, con_i \rangle \in C$ such that $p_i = def_i(s \downarrow_{con_i})$ and $\exists \langle def_j, con_j \rangle \in C$ such that $n_j = def(s \downarrow_{con_j})$. Hence the preference of a solution is obtained by combining all the positive preferences associated to its projections over the constraints on one side, all the negative preferences associated to its projections over the constraints on the other side, and then compensating the two preferences so obtained. This definition is in accordance with the classical tool used in bipolar decision making, namely with cumulative prospect theory [27].

A solution s is *optimal* if there is no other solution s' with $ovpref(s') > ovpref(s)$.

Given a bipolar constraint $c = \langle def, con \rangle$ and one of its tuple t, it is possible to define two functions pos and neg as follows:

$$pos(c)(t) = \begin{cases} def(t) & \text{if } def(t) \in P, \\ \square & \text{otherwise,} \end{cases}$$

$$neg(c)(t) = \begin{cases} def(t) & \text{if } def(t) \in N, \\ \square & \text{otherwise.} \end{cases}$$

In other words, given a constraint c and one of its tuple t, $pos(c)(t)$ (resp., $neg(c)(t)$) returns the preference given by c for the tuple t if it is positive (resp., negative) and indifference otherwise.

Example 1. Figure 1 shows an example of a BCSP. It is defined on the same bipolar preference structure considered before, that is, $\langle N = [-1, 0], P = [0, 1], + = max, \times, \bot = -1, \Box = 0, \top = 1\rangle$, where $\times$ is s.t. $\times_p = max$, $\times_n = min$ and $\times_{np} = sum$. It is composed by four variables, that is, x, y, z_1 and z_2, and by the three bipolar constraints $\langle q, \{x, z_1\}\rangle$, $\langle t, \{x, z_2\}\rangle$ and $\langle f, \{x, y\}\rangle$. The domain of x and y is $\{a, b\}$, while the domain of z_1 and z_2 is $\{a, b, c\}$. One of the solutions of such a BCSP is $s = (y = b, x = a, z_1 = a, z_2 = b)$. To compute its preference, we must consider the preferences of all the projections of s in the various constraints, i.e., the preference $+0.8$ of $(y = b, x = a)$ in $\langle f, \{x, y\}\rangle$, the preference -0.5 of $(x = a, z_1 = a)$ in $\langle q, \{x, z_1\}\rangle$, and the preference -0.3 of $(x = a, z_2 = b)$ in $\langle t, \{x, z_2\}\rangle$. Thus $ovpref(s) = (-0.5 \times_n -0.3) \times_{np} 0.8 = min(-0.5, -0, 3) + 0.8 = -0.5 + 0.8 = +0.3$. In this example an optimal solution is $s' = (y = b, x = a, z_1 = b, z_2 = c)$ with preference $ovpref(s') = +0.8$. Let us now show how functions pos and neg defined above work on the constraint $c_1 = \langle q, \{x, z_1\}\rangle$ and on the tuples $t_1 = (x = a, z_1 = a)$ and $t_2 = (x = a, z_1 = b)$. For t_1 we have $pos(c_1)(t_1) = 0$ and $neg(c_1)(t_1) = -0.5$, and for t_2 we have $pos(c_1)(t_2) = +0.8$ and $neg(c_1)(t_2) = 0$. $\Box$

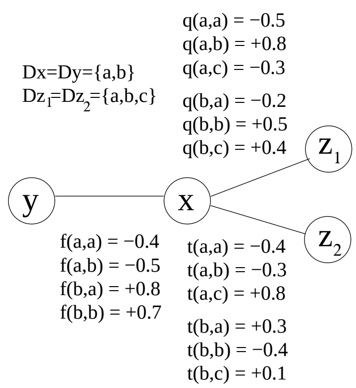

Fig. 1. A BCSP.

2.5 Uncertainty in soft constraint problems

Uncertain soft constraint satisfaction problems (USCSPs) [26, 25] are soft constraint problems where some variables are uncontrollable, i.e., they are not under the user's control. They can model many real-life problems, such as scheduling and timetabling. For example, they can model the problem of scheduling some tasks, knowing that the duration of some of those is uncertain, and only vaguely known [15], or the problem of deciding how many training sessions to perform in a tutorial, without knowing the effective number of participants, but knowing only an approximately number of these participants [16]. Contrarily to classical constraint problems, in USCSPs we cannot decide how to assign the variables to make the assignment optimal, but we must assign values to the controllable variables, denoted with V_c, guessing what Nature will do with the uncontrollable variables, denoted with V_u.

If the uncontrollable variables are equipped with additional information on the likelihood of their values, like in our case, such an information can be used to infer new soft constraints over the controllable variables, which express the compatibility of the controllable part of the problem with the uncontrollable one. This information can be used to define the notion of optimal solution. It is assumed that there is no observability over uncertain events before decision.

An USCSP is thus defined as a set of variables, which can be controllable or uncontrollable, and a set of soft constraints over these variables. Moreover, the domain of every uncontrollable variable is equipped with a possibility distribution, that specifies, for every value in the domain, the degree of plausibility that the variable takes that value.

More formally, a *possibility distribution* π associated to a variable z with domain A_Z is a mapping from A_Z to a totally ordered scale L (usually $[0, 1]$) such that $\forall a \in A_Z$, $\pi(a) \in L$ and $\exists\ a \in A_Z$ such that $\pi(a) = \mathbf{1}$, where $\mathbf{1}$ is the top element of the scale L [28].

An *uncertain soft constraint satisfaction problem* (USCSP) is a tuple $\langle S, V_c, V_u, \pi, C_c, C_{cu}, C_u\rangle$ where S is a c-semiring, $V_c = \{x_1, \ldots x_n\}$ is a set of controllable variables, $V_u = \{z_1, \ldots z_k\}$ is a set of uncontrollable variables, $\pi = \{\pi_1, \ldots, \pi_k\}$ is a set of possibility distributions over V_u, such that every $z_i \in Vu$ has possibility distribution π_i with scale $[0, 1]$, C_c is the set of constraints that involve only variables of V_c, C_{cu} is a set of constraints that involve at least a variable in V_c and a variable in V_u, and that may involve any other variable of $V_c \cup V_u$, and C_u is the set of constraints that involve only variables of V_u.

Notice that when the set of uncontrollable variables, i.e., V_u, is empty, then the sets of constraints involving variables in V_u, i.e., C_{cu} and C_u, are empty, and the USCSP corresponds to a soft constraint problem $\langle S, V_c, C_c\rangle$, as defined in Section 2.1.

When the chosen semiring is $S_{FCSP} = \langle [0, 1], max, min, 0, 1\rangle$, the definition of an USCSP models an Uncertain Fuzzy CSP (UFCSP), that corresponds, when there are no uncontrollable variables, to an FCSP, as defined in Section 2.1.

Example 2. Figure 2 shows an example of an UFCSP. Each constraint is defined by associating a preference level (in this case between 0 and 1) to each assignment of its variables to values in their domains. The set V_c of the controllable variables is composed by x, y, and w, while the set V_u of the uncontrollable variables contains only z. The values in the domain of z are characterized by the possibility distribution π_Z. The set of constraints C_c is composed by the constraint $\langle q, \{x, w\}\rangle$, which relates x and w via the preference function q. The set of constraints C_{cu} is composed by the constraint $\langle f, \{x, y, z\}\rangle$, which is defined on variables x, y, and z by the preference function f, while the set C_u is empty. □

Given an assignment t to all the variables of an USCSP, its overall preference is computed by combining, via the $\times$ operator, the preference levels of its subtuples in the selected constraints. More formally, given an USCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu}, C_u\rangle$, let t be an assignment to all the variables of Q, then its *overall preference* is the value $ovpref(t) = \prod_{\{\langle def_i, con_i\rangle \in C_c \cup C_{cu} \cup C_u\}} def_i(t \downarrow_{con_i})$.

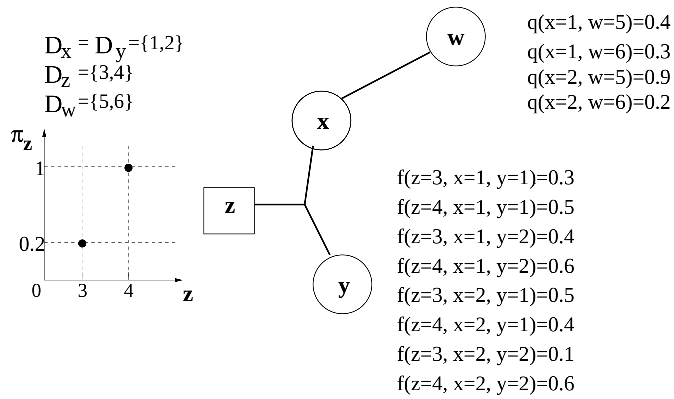

Fig. 2. An UFCSP.

A solution of an USCSP is a complete assignment to all its controllable variables. More formally, given an USCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu}, C_u\rangle$, a *solution* of Q is a complete assignment to all the variables of V_c.

2.6 Preference, robustness, and desirable properties for USCSPs

In [26], a solution s of an USCSP is associated to both a preference degree, written $pref(s)$, and a degree of robustness, written $rob(s)$. The preference degree summarizes all the preferences in the controllable part and it can be really obtained for some assignment to the uncontrollable variables decided by the Nature. The robustness of a solution, that measures what is the impact of Nature on the preference obtained by choosing that solution, is assumed to be dependent both on the preferences in the constraints connecting both controllable and uncontrollable variables to s and on such possibility distributions.

Two desirable properties for the notion of robustness that have been considered in [25, 26] for USCSPs and in [17] for UFCSPs are the following.

Property P1. *Given solutions s and s' of an USCSP, $\langle S, V_c, \pi, V_u, C_c, C_{cu}\rangle$, where every variable v_i in V_u is associated to a possibility distribution π_i, if for every constraint $\langle def, con\rangle \in C_{cu}$ and for every assignment a to the uncontrollable variables in con, $def((s,a)\downarrow_{con}) \leq_S def((s',a)\downarrow_{con})$, then it should be that $rob(s) \leq_S rob(s')$.*

In other words, if we increase the preferences of any tuple involving uncontrollable variables, solution should have a higher value of robustness.

Property P2. *Take a solution s of the USCSPs $Q_1 = \langle S, V_c, V_u, \pi_1, C_c, C_{cu}\rangle$ and $Q_2 = \langle S, V_c, V_u, \pi_2, C_c, C_{cu}\rangle$. Assume for every assignment a to variables in V_u, $\pi_2(a) \leq \pi_1(a)$. Then it should be that $rob_{\pi_1}(s) \leq_S rob_{\pi_2}(s)$, where rob_{π_1} is the robustness computed in the problem Q_1, and rob_{π_2} is the robustness computed in the* *problem* Q_2.

In other words, if we lower the possibility of any value of the uncontrollable variables, solution should have a higher value of robustness.

To understand which solutions are better than others in an USCSP, in [26] it is considered a solution ordering, say $\succ$, which is reflexive and transitive that should depend on the notions of robustness and preference as follows [26]:

Property P3. *Given two solutions* s *and* s' *of an USCSP, if* $rob(s) = rob(s')$ *and* $pref(s) >_S pref(s')$*, it should be that* $s \succ s'$.

Property P4. *Given two solutions* s *and* s' *of an USCSP such that* $pref(s) = pref(s')$*, and* $rob(s) >_S rob(s')$*, then it should be that* $s \succ s'$.

In other words, two solutions which are equally good with respect to one aspect (robustness or preference degree) and differ on the other should be ordered according to the discriminating aspect.

Property P5. *Given two solutions* s *and* s'*, an USCSP* $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu} \rangle$*, such that* $ovpref(s, a) >_S ovpref(s', a)$*,* $\forall a$ *assignment to* V_u*, then it should be that* $s \succ s'$.

In other words, if two solutions s and s' are such that the overall preference of the assignment (s, a) to all the variables is better than or equal to one of (s', a) for all the values a of the uncontrollable variables, then s should be considered better than the other one.

2.7 Removing uncertainty in UFCSPs: preferences, robustness and semantics

In [26] a method is presented to remove uncontrollable variables from uncertain fuzzy CSPs preserving as much information as possible. Starting from this method, both a degree of preference and a degree of robustness for a solution are defined, and it is shown that these degrees satisfy the desirable properties mentioned above.

Removing uncertainty. The procedure presented in [26] to remove uncertainty in UFCSPs, that is called Algorithm *SP*, works as follows. It takes as input an UFCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu} \rangle$, where every variable $z_i \in V_u$ has a possibility distribution π_i and where S is the fuzzy c-semiring and returns an RFCSP that is similar to an FCSP but has two sets of constraints rather than one. More precisely, an RFCSP is a tuple $\langle S, V_c, C_1, C_2 \rangle$ such that $\langle S, V_c, C \rangle$, where $C = C_1 \cup C_2$, is an FCSP.

The RFCSP Q' returned by *SP* is obtained from Q by eliminating its uncontrollable variables and the fuzzy constraints in C_{cu} relating controllable and uncontrollable variables, and by adding new fuzzy constraints only among these controllable variables that we call C_{proj} (*the fuzzy projection constraints*) and C_{rob}(*the fuzzy robustness constraints*), that encode (some of) the information contained in the uncontrollable part

Algorithm 1: *SP*

Input: $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu} \rangle$: an UFCSP;
Output: $Q' = \langle S, V_c, C_c^*, C_{rob} \rangle$: an RFCSP;
$C_{rob} \leftarrow \emptyset$;
$C_{proj} \leftarrow \emptyset$;
foreach *constraint* $c \in C_{cu}$ **do**
    $C_{rob} \leftarrow C_{rob} \cup ComputeFuzzyRobustnessConstraint(c)$;
    $C_{proj} \leftarrow C_{proj} \cup ComputeFuzzyProjectionConstraint(c)$;
$C_c^* \leftarrow C_c \cup C_{proj}$;
$Q' \leftarrow \langle S, V_c, C_c^*, C_{rob} \rangle$;
return Q';

of the problem. In particular, it adds C_{proj} to C_c, while it keeps C_{rob} separate. More precisely, given a constraint $c = \langle def, con \rangle$ in C_{cu} such that $con \cap V_c = X$ and $con \cap V_u = Z$,

– its corresponding robustness contraint in C_{rob}, obtained by applying the procedure *ComputeFuzzyRobustnessConstraint(c)*, returns a fuzzy constraint $\langle def', X \rangle$ where, $\forall t_X$ assignment to X,

$$def'(t_X) = min_{t_Z \in A_Z} max(def(t_X, t_Z), 1 - \pi_Z(t_Z))).$$

– its corresponding projection constraint in C_{proj}, obtained by applying the procedure *ComputeFuzzyProjectionConstraint(c)*, is the constraint $\langle def'', X \rangle$, where

$$def''(t_X) = max_{\{a \in A_Z\} | \pi_Z(a) > 0} def(t_X, t_Z).$$

Preference, robustness and semantics in UFCSPs. In [26] the problem returned by the algorithm *SP* is used to define the preference and the robustness of a solution in an UFCSP. More precisely, given a solution s of an UFCSP Q, let $Q' = \langle S, V_c, C_c^*, C_{rob} \rangle$, where $C_c^* = C_c \cup C_{proj}$, the RFCSP obtained from Q by algorithm *SP*,

– the *preference* of s is $pref(s) = min_{\{\langle def, con \rangle \in C_c^*\}} def(s \downarrow_{con})$
– the robustness of s is $rob(s) = min_{\{\langle def, con \rangle \in C_{rob}\}} def(s \downarrow_{con})$.

In other words, the preference (resp., robustness) of a solution is obtained by combining the preferences of the appropriate subtuples of the solution over the constraints in C_c^*, i.e., in $C_c \cup C_{proj}$ (resp., in C_{rob}). In [26] it is shown that the desirable properties on the robustness (i.e., Property P1 and Property P2) presented previously hold.

Since a solution of an UFCSP is associated to a preference and a robustness degree, in [26] various semantics are defined to order the solutions which depend on the attitude w.r.t. these two notions. In the following we will describe those that we will consider in this paper.

– *Risky* semantics: given $A1 = (pref_1, rob_1)$ and $A2 = (pref_2, rob_2)$, $A1 \succ_{Risky} A2$ iff $pref_1 > pref_2$ or ($pref_1 = pref_2$ and $rob_1 > rob_2$). Informally, the idea is to give more relevance to the preference that can be reached in the best case considering less important a high risk of being inconsistent.

- *Safe* semantics: given $A1 = (pref_1, rob_1)$ and $A2 = (pref_2, rob_2)$, $A1 \succ_{Safe} A2$ iff $rob_1 > rob_2$ or ($rob_1 = rob_2$ and $pref_1 > pref_2$). The idea is to give more importance to the robustness level that can be reached considering less important having a high preference.
- *Diplomatic* semantics: given $A1 = (pref_1, rob_1)$ and $A2 = (pref_2, rob_2)$, $A1 \succ_{Dipl} A2$ iff ($pref_1 \geq pref_2$ and $rob_1 \geq rob_2$) and ($pref_1 > pref_2$ or $rob_1 > rob_2$). The idea is that a pair is to be preferred to another only if it wins both on preference and robustness, leaving incomparable all the pairs that have one component higher and the other lower.

In [26] it is shown that for Risky, Safe and Diplomatic semantics the desired properties on solution ordering (i.e., Properties P3 and P4) presented previously hold. Also, they prove that Property P5 is satisfied only by $\succ_{Risky}$.

3 Uncertain bipolar problems

Uncertain bipolar problems (UBCSPs) are characterized by a set of variables, each of which can be controllable or uncontrollable, and by a set of bipolar constraints. Thus, an UBCSP is a BCSP where some of the variables are uncontrollable. Moreover, the domain of every uncontrollable variable is equipped with a possibility distribution, that specifies, for every value in the domain, the degree of plausibility that the variable takes that value. Hence, an UBCSP is also an USCSP where every constraint is bipolar. More formally,

Definition 1 (UBCSP). *An uncertain bipolar CSP is a tuple $\langle S, V_c, V_u, \pi, C_c, C_{cu} \rangle$, where*

- $S = (N, P, +, \times, \bot, \Box, \top)$ *is a bipolar preference structure and* $\leq_S$ *is the ordering induced by operator* $+$*;*
- $V_c = \{x_1, \dots x_n\}$ *is a set of controllable variables;*
- $V_u = \{z_1, \dots z_k\}$ *is a set of uncontrollable variables;*
- $\pi = \{p_1, \dots p_k\}$ *is a set of possibility distributions over* V_u*. In particular, every* $z_i \in V_u$ *has possibility distribution* π_i *with scale* $[0, 1]$*;*
- C_c *is the set of bipolar constraints that involve only variables of* V_c*;*
- C_{cu} *is a set of bipolar constraints that involve at least a variable in* V_c *and a variable in* V_u *and that may involve any other variable of* $(V_c \cup V_u)$*.*
- C_u *is the set of bipolar constraints that involve only variables of* V_u*.*

For simplicity we will assume that C_u is empty and thus we will omit it in the tuple when we refer to an UBCSP. If $C_u \neq \emptyset$, we can translate every constraint of type C_u in a new constraint of type C_{cu}, thus obtaining an UBCSP with $C_u = \emptyset$. This can be done by using a procedure similar to the one used for UFCSPs in [26].

Given an assignment t to all the variables of an UBCSP, its overall preference (see Section 2) is computed by combining, via the $\times$ operator, first all the positive preferences of its subtuples in the selected constraints, then all the negative preferences of its subtuples in the selected constraints, and finally the two resulting preferences. More formally, using the notation presented in this section,

Definition 2 (overall assignment preference). *Given an UBCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$, let t be an assignment to all the variables of Q, then its overall preference is the value $ovpref(t) = ovpref_p(t) \times ovpref_n(t)$, where $ovpref_p(t) = \prod_{\{\langle def_i, con_i\rangle \in C_c \cup C_{cu}\}} pos(def_i)(t \downarrow_{con_i})$, and $ovpref_n(t) = \prod_{\{\langle def_i, con_i\rangle \in C_c \cup C_{cu}\}} neg(def_i)(t \downarrow_{con_i})$.*

A solution of an UBCSP is a complete assignment to all its controllable variables. More formally,

Definition 3 (solution). *Given an UBCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$, a solution of Q is a complete assignment to all the variables of V_c.*

Example 3. An example of an UBCSP is the one presented in Figure 3 (a). It is like the one in Figure 1, except that now variables z_1 and z_2 are uncontrollable and characterized by two possibility distributions π_1 and π_2. More formally, such an UBCSP is defined by the tuple $\langle S, V_c = \{x, y\}, V_u = \{z_1, z_2\}, \pi = \{pi_1, \pi_2\}, C_c, C_{cu}\}\rangle$. We recall that the bipolar structure is $\langle N = [-1, 0], P = [0, 1], + = max, \times, \bot = -1, \Box = 0, \top = 1\rangle$, where $\times$ is s.t. $\times_p = max$, $\times_n = min$ and $\times_{np} = sum$. The set of constraints C_c contains $\langle f, \{x, y\}\rangle$, while C_{cu} contains $\langle q, \{x, z_1\}\rangle$ and $\langle t, \{x, z_2\}\rangle$. Figure 3 (a) shows the positive and the negative preferences within such constraints and the possibility distributions π_1 and π_2 over the domains of z_1 and z_2. $\Box$

4 Preference, robustness, and desirable properties in UBCSPs

Given a solution s of an UBCSP, we will associate to it a degree of preference, say $pref(s)$, and a degree of robustness, say $rob(s)$ that generalize those given for USCSPs in [26]. Moreover, we will show that these notions satisfy the following generalized version of the desirable properties for USCSPs described in Section 2.6:

Property BP1. *Given solutions s and s' of an UBCSP, $\langle S, V_c, \pi, V_u, C_c, C_{cu}\rangle$, where every variable v_i in V_u is associated to a possibility distribution π_i, if for every constraint $\langle def, con\rangle \in C_{cu}$ and for every assignment a to the uncontrollable variables in con, $def((s, a) \downarrow_{con}) \leq_S def((s', a) \downarrow_{con})$, then it should be that $rob(s) \leq_S rob(s')$.*

Property BP2. *Take a solution s of the UBCSPs $Q_1 = \langle S, V_c, V_u, \pi_1, C_c, C_{cu}\rangle$ and $Q_2 = \langle S, V_c, V_u, \pi_2, C_c, C_{cu}\rangle$. Assume for every assignment a to variables in V_u, $\pi_2(a) \leq \pi_1(a)$. Then it should be that $rob_{\pi_1}(s) \leq_S rob_{\pi_2}(s)$, where rob_{π_1} is the robustness computed in the problem Q_1, and rob_{π_2} is the robustness computed in the problem Q_2.*

Property BP3. *Given two solutions s and s' of an UBCSP, if $rob(s) = rob(s')$ and $pref(s) >_S pref(s')$, it should be that $s \succ s'$.*

Property BP4. *Given two solutions s and s' of an UBCSP such that $pref(s) = pref(s')$, and $rob(s) >_S rob(s')$, then it should be that $s \succ s'$.*

Property BP5. *Given two solutions s and s', an UBCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$,* *such that* $ovpref_p(s,a) >_S ovpref_p(s',a)$ *and* $ovpref_n(s,a) >_S ovpref_n(s',a)$ $\forall a$ *assignment to* V_u*, then it should be that* $s \succ s'$

Notice that the new desirable properties for bipolar preferences are similar to the ones given for USCSPs in [26]. Two differences are that they consider an UBSCP rather than an USCSP and that, as preference ordering, they consider $\leq_S$, that is the ordering induced by the additive operator of the bipolar preference structure of the considered UBSCP and not the ordering induced by the additive operator of the c-semiring of the considered USCSP. Another difference is in Property BP5 where not only the negative overall preferences are considered (as in USCSPs), but also the positive overall preferences.

5 Removing uncertainty from UBCSPs over closed real intervals

We now show how to extend the approach shown in [25, 26] to deal with UFCSPs, i.e., problems with fuzzy preferences and uncertainty (see Section 2.7), to UBCSPs over real intervals, i.e., problems with bipolar preferences and uncertainty where the set of the positive preferences and the set of the negative preferences are two closed intervals of $\mathbb{R}$ (or structures isomorph to it)[2]. Starting from the generalization of this approach, we will define robustness and preference degrees and we will show that they satisfy the properties which are considered desirable (see Theorems 1, 2, 3, 4, and 5).

Our procedure, that we call Algorithm *B-SP*, takes as input an UBCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu} \rangle$, where every variable $z_i \in V_u$ has a possibility distribution π_i and $S = \langle N, P, +, \times, \bot, \Box, \top \rangle$ is any bipolar preference structure where N and P are two closed intervals of $\mathbb{R}$ (or structures isomorph to them) and it returns an RBCSP. We recall that an RBCSP is similar to a BCSP but has two sets of constraints rather than one (see Section 2.4).

Algorithm 2: *B-SP*

Input: $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu} \rangle$: an UBCSP;
Output: $Q' = \langle S, V_c, C_c^*, C_{rob} \rangle$: an RBCSP;
$C_{rob} \leftarrow \emptyset$;
$C_{proj} \leftarrow \emptyset$;
foreach *constraint* $c \in C_{cu}$ **do**
    $C_{rob} \leftarrow C_{rob} \cup ComputeRobustnessConstraint(c)$;
    $C_{proj} \leftarrow C_{proj} \cup ComputeProjectionConstraint(c)$;
$C_c^* \leftarrow C_c \cup C_{proj}$;
$Q' \leftarrow \langle S, V_c, C_c^*, C_{rob} \rangle$;
return Q';

[2] Notice that the procedure that we propose holds also for intervals of $\mathbb{Q}$, and it can be easily adapted also to handle closed intervals of $\mathbb{Z}$.

The RBCSP Q' returned by *B-SP* is obtained from Q by eliminating its uncontrollable variables and the bipolar constraints in C_{cu} relating controllable and uncontrollable variables, and by adding new bipolar constraints only among these controllable variables that we call C_{proj} and C_{rob}. In particular, it adds C_{proj} to C_c, while it keeps C_{rob} separate. More precisely, C_{proj} (*the projection constraints*) is the set of constraints obtained applying to every constraint c in C_{cu} of Q the procedure *ComputeProjection-Constraint(c)*, that will be described in Section 5.2, while C_{rob} (*the robustness constraints*) is the set of constraints obtained applying to every constraint c in C_{cu} of Q the procedure *ComputeRobustnessConstraint(c)*, that will be described in Section 5.1. In Sections 5.1 and 5.2 we will see that these new constraints will encode (some of) the information contained in the uncontrollable part of the problem.

As in [26] for the fuzzy case, starting from this problem Q', we define the preference degree of a solution considering the preference functions of the constraints in $C_c \cup C_{proj}$, and the robustness degree of a solution considering the preference functions of the constraints in C_{rob}.

Notice that Algorithm *B-SP* is similar to Algorithm *SP* [26] described in Section 2.7. However, it takes in input an UBCSP rather than an UFCSP, it returns an RBCSP rather than an RFCSP, and it uses different procedures for computing robustness and projection constraints that depend on specific properties of the bipolar preference structure of the considered UBCSP.

5.1 Robustness constraints

Similarly to the approach for the fuzzy case [26], the set of robustness constraints C_{rob} is composed by the bipolar constraints obtained by reasoning on preference functions of the bipolar constraints in C_{cu} and on the possibilities associated to values in the domains of uncontrollable variables involved in such constraints. However, the procedure to obtain such bipolar constraints is different from the one considered in the fuzzy case, since while in the fuzzy case it is exploited the fact that fuzzy preferences and possibilities are commensurable, in the bipolar context we cannot exploit this fact since positive and negative preferences may not be commensurable with possibilities. We have thus adapted the fuzzy approach used to defined robustness constraints to take into account this fact.

More precisely, every constraint in C_{rob} is built by exploiting the procedure denoted *ComputeRobustnessConstraint* in algorithm *B-SP*, that works as follows.

- (*Normalization*) Every constraint $c = \langle def, con \rangle$ in C_{cu} such that $con \cap V_c = X$ and $con \cap V_u = Z$, is translated in two bipolar constraints $\langle defp, con \rangle$ and $\langle defn, con \rangle$, with preferences in $[0, 1]$, where, $\forall(t_X, t_Z)$ assignment to $X \times Z$,

  $$defp(t_X, t_Z) = g_p(pos(c)(t_X, t_Z))$$

  and $defn(t_X, t_Z) = g_n(neg(c)(t_X, t_Z))$. If the positive (resp., negative) preferences are defined in the interval of $\mathbb{R}$, $P = [a_p, b_p]$ (resp., $N = [a_n, b_n]$) then g_p: $[a_p, b_p] \to [0, 1]$ (resp., g_n: $[a_n, b_n] \to [0, 1]$) associates to every $x \in [a_p, b_p]$ the value $\frac{x-a_p}{b_p-a_p} \in [0, 1]$ (resp., to every $x \in [a_n, b_n]$ the value $\frac{x-a_n}{b_n-a_n}$) by using the classical division and subtraction operation of $\mathbb{R}$.

– (*Removing uncontrollability*) The constraint $\langle defp, con\rangle$ obtained before is then translated in $\langle defp', X\rangle$, and $\langle defn, con\rangle$ is then translated in $\langle defn', X\rangle$, where, $\forall t_X$ assignment to X,

$$defp'(t_X) = \inf_{t_Z \in A_Z} \sup(defp(t_X, t_Z), c_S(\pi_Z(t_Z))),$$

and $defn'(t_X) = \inf_{t_Z \in A_Z} \sup(defn(t_X, t_Z), c_S(\pi_Z(t_Z)))$, where c_S is an order reversing map with respect to $\leq_S$ in $[0, 1]$, such that $c_S(c_S(p)) = p$ and $\inf$ which is the opposite of the $\sup$ operator (derived from operator $+$ of S), applied to a set of preferences, returns its worst preference with respect to the ordering $\leq_S$.

– (*Denormalization*) The constraint $\langle defp', X\rangle$ obtained before is then translated in $\langle defp'', X\rangle$, and $\langle defn', X\rangle$ is then translated in $\langle defn'', X\rangle$, where $\forall t_X$ assignment to X,

$$defp''(t_X) = g_p^{-1}(defp'(t_X)),$$

and $defn''(t_X) = g_n^{-1}(defn'(t_X))$. The map g_p^{-1}:$[0, 1] \to [a_p, b_p]$ associates to every $y \in [0, 1]$ the value $[y(b_p - a_p) + a_p] \in [a_p, b_p]$, and the map g_n^{-1}:$[0, 1] \to [a_n, b_n]$ associates to every $y \in [0, 1]$ the value $[y(b_n - a_n) + a_n] \in [a_n, b_n]$.

Hence, given $c = \langle def, X \cup Z\rangle \in C_{cu}$, its corresponding robustness constraints in C_{rob} are the bipolar constraints $\langle defp'', X\rangle$ and $\langle defn'', X\rangle$ defined above. When we compute the robutsness constraints, we reason separately on positive and negative preferences since in our approach commensurability with possibilities applies only separately to positive and negative preference sets, and not to the whole preference set. Forcing the commensurability of the possibility range with the bipolar preference set would induce a bipolarization of possibilities, which is not reasonable. However, in order to avoid loss of information, when we compute the robustness degree of a solution, considering the robustness constraints, we compensate the positive and the negative preferences of such constraints.

It is possible to show that the functions g_p and g_n are strictly monotonic with respect to the ordering $\leq_S$ induced by the operator $+$ of S. Hence such functions are invertible and their inverse functions are monotonic with respect to the same ordering.

Proposition 1. *Given* $a_p, b_p, a_n, b_n \in \mathbb{R}$, *with* $a_p <_S b_p$ *and* $a_n <_S b_n$ *the following maps are strictly monotone w.r.t. the ordering induced* $\leq_S$*:* g_p*:* $[a_p, b_p] \to [0, 1]$ *s.t.* $x \mapsto \frac{x-a_p}{b_p-a_p}$*, and* g_n*:* $[a_n, b_n] \to [0, 1]$ *s.t.* $x \mapsto \frac{x-a_n}{b_n-a_n}$.

Proof. We now show that g_p is monotone w.r.t. $\leq_S$. If $x_1 >_S x_2$, then $x_1 - a_p >_S x_2 - a_p$, by monotonicity of the subtraction among real numbers. Moreover, since $b_p >_S a_p$, then $b_p - a_p >_S 0$ and also $\frac{1}{b_p-a_p} > 0$. Thus, by strict monotonicity of the product over real numbers ($\forall a, b, c \in \mathbb{R}$, if $c >_S 0$ and $a >_S b$, then $ac >_S bc$), $\frac{x_1-a_p}{b_p-a_p} >_S \frac{x_2-a_p}{b_p-a_p}$, i.e., $g_p(x_1) >_S g_p(x_2)$. Similarly, since $b_n >_S a_n$, and thus $\frac{1}{b_n-a_n} > 0$, it is possible to show that g_n is strictly monotone. □

This allows to show that the new preference functions $defp''$ and $defn''$ in the constraints C_{rob} satisfy the same property given in [17, 25]. That is, given an assignment t_X to controllable variables in X in a constraint $c = \langle def, con\rangle \in C_{cu}$, where

$con = X \cup Z$, the higher are $defp''(t_X)$ and $defn''(t_X)$, the more assignments to uncontrollable variables in c will yield in Q preference higher than a given threshold. It is thus possible to prove that:

– $defp''(d) \geq_S \beta \in P$ (resp., $defn''(d) \geq_S \beta \in N$) if and only if, for any t_Z assignment to Z with $\pi_Z(t_Z) > c_S(g_p(\beta))$ (resp., $\pi_Z(t_Z) > c_S(g_n(\beta))$), then $def(t_X, t_Z) \geq_S \beta$.

Note that this property holds both for positive and negative preferences, since the definition of $defp''$ and $defn''$ it is not based on the combination operators ($\times_p$ and $\times_n$) of positive and negative preferences, which have different behaviours, but only on the operators $\sup$ and $\inf$ derived by the additive operators $+_p$ and $+_n$, which satisfy the same properties. More precisely,

Proposition 2. *Consider an UBCSP $\langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$, where $S = \langle N, P, +, \times, \perp, \Box, \top\rangle$ is a bipolar preference structure where $P = [a_p, b_p]$ and $N = [a_n, b_n]$ are closed intervals of $\mathbb{R}$. For every constraint $c = \langle def, con\rangle \in C_{cu}$ such that $con \cap V_u = Z$, with possibility distribution π_Z, and $con \cap V_u = X$, the corresponding robustness constraints $\langle defp'', X\rangle$ and $\langle defn'', X\rangle$ are such that, for every t_X assignment to X,*

– $defp''(t_X) \geq_S \beta \in P$ *iff, when* $\pi_Z(t_Z) > c_S(g_p(\beta))$, *then* $pos(c)(t_X, t_Z) \geq_S \beta$,
– $defn''(t_X) \geq_S \alpha \in N$ *iff, when* $\pi_Z(t_Z) > c_S(g_n(\alpha))$, *then* $pos(c)(t_X, t_Z) \geq_S \alpha$,

where t_Z is an assignment to Z, g_p: $[a_p, b_p] \rightarrow [0, 1]$ is such that $x \mapsto \frac{x-a_p}{b_p-a_p} \in [0, 1]$ g_n: $[a_n, b_n] \rightarrow [0, 1]$ is such that $x \mapsto \frac{x-a_n}{b_n-a_n}$, and c_S is an order reversing map with respect to ordering $\leq_S$ in $[0, 1]$ such that $c_S(c_S(p)) = p$, $\forall p \in [0, 1]$.

Proof. We show the first statement concerning $defp''(t_X)$. The second one, concerning $defn''(t_X)$, can be proved analogously, since by construction g_n and g_n^{-1} have the same properties respectively of g_p and g_p^{-1}. We recall that $defp''(t_X) = g_p^{-1}(inf_{t_Z \in A_Z}$ $(g_p(pos(c)(t_X, t_Z)) + c_S(\pi_Z(t_Z))))$, where A_Z is the set of the assignment to Z.
($\Rightarrow$) We assume that $defp''(t_X) \geq_S \beta$. If this holds, then, since g_p is monotone with respect to the ordering $\leq_S$, $g_p(defp''(t_X)) \geq_S g_p(\beta)$, i.e., $g_p(g_p^{-1}(inf_{t_Z \in A_Z}$ $sup(g_p(pos(c)(t_X, t_Z)), c_S(\pi_Z(t_Z))))) \geq_S g_p(\beta)$, that is, since g_p is the inverse function of g_p^{-1}, $inf_{t_Z \in A_Z}\ sup(g_p(pos(c)(t_X, t_Z)),\ c_S(\pi_Z(t_Z)))) \geq_S g_p(\beta)$. Since we are considering totally ordered preferences, this implies that $sup(g_p(pos(c)(t_X, t_Z))$, $c_S(\pi_Z(t_Z))) \geq_S g_p(\beta)$, $\forall t_Z \in A_Z$. For t_Z with $\pi_Z(t_Z) > c_S(g_p(\beta))$, since c_S is an order reversing map with respect to $\leq_S$ such that $c_S(c_S(p)) = p$, we have $c_S(\pi_Z(t_Z))$ $<_S c_S(c_S(g_p(\beta)) = g_p(\beta)$. Therefore, for such a value t_Z, we have that $g_p(pos(c)(t_X,$ $t_Z)) = sup\ (g_p(pos(c)(t_X, t_Z)), c_S(\pi_Z(t_Z))) \geq_S g_p(\beta)$ and, since g_p^{-1} is monotone, we have $g_p^{-1}(g_p(pos(c)(t_X, t_Z))) \geq_S g_p^{-1}(g_p(\beta))$, i.e., $pos(c)(t_X, t_Z) \geq_S \beta$.
($\Leftarrow$) We assume that $\forall t_Z$ with $\pi_Z(t_Z) > c_S(g_p(\beta))$, $pos(c)(t_X, t_Z) \geq_S \beta$. Then, for such t_Z, since g_p is monotone with respect to $\leq_S$, $g_p(pos(c)(t_X, t_Z)) \geq_S g_p(\beta)$ and so, $sup(g_p(pos(c)(t_X, t_Z)), c_S(\pi_Z(t_Z))) \geq_S g_p(\beta)$. On the other hand, for every t_Z such that $\pi_Z(t_Z) < c_S(g_p(\beta))$, we have $c_S(\pi_Z(t_Z)) >_S g_p(\beta)$ and so $sup(g_p(pos(c)(t_X,$ $t_Z)), c_S(\pi_Z(t_Z))) >_S g_p(\beta)$. Thus $\forall t_Z \in A_Z$, $sup(g_p(pos(c)(t_X, t_Z)), c_S(\pi_Z(t_Z)))$ $\geq_S g_p(\beta)$ and so $inf_{t_Z \in A_Z} sup(pos(c)(t_X, t_Z), c_S(\pi_Z(t_Z))) \geq_S g_p(\beta)$. Hence, since g_p^{-1} is monotone, $g_p^{-1}(inf_{t_Z \in A_Z}\ (sup(pos(c)(t_X, t_Z),\ c_S(\pi_Z(t_Z))))) \geq_S g_p^{-1}(g_p(\beta))$,

i.e., $defp''(t_X) \geq_S \beta$. □

Example 4. Consider the constraint $c_1 = \langle q, \{x, z_1\}\rangle$ in Figure 3 (a). The robustness constraints obtained from it are the constraints $r1 = \langle qp'', \{x\}\rangle$ and $r2 = \langle qn'', \{x\}\rangle$ shown in Figure 3 (b). They have been obtained by assuming g_p the identity map, $g_n : N = [-1, 0] \to [0, 1]$ mapping every value $n \in [-1, 0]$ into the value $(n + 1) \in [0, 1]$, $g_n^{-1} : [0, 1] \to [-1, 0]$ mapping every value $t \in [0, 1]$ into the value $(t - 1) \in [-1, 0]$, and c_S mapping every $p \in [0, 1]$ in $1-p$. We now show the meaning of these robustness constraints. The value $qp''(x = a) = 0.3$ means that in c_1, as shown by the property above, for all the values t_i of z_1 with possibility $\pi_1(t_i) > 1 - 0.3 = 0.7$, (in this case only b), we have $q(x = a, t_i) \geq 0.3$. Analogously, the value $qn''(x = a) = -0.5$ means that, for all the values t_i of z_1 with possibility $\pi_1(t_i) > 1 - (-0.5 + 1) = 0.5$, (that is, for a, b, and c), we have in c_1 that $q(x = a, t_i) \geq -0.5$. □

5.2 Projection constraints

As in the approach for the fuzzy case [26], projection constraints are added to the problem in order to recall part of the information contained in the constraints in C_{cu} that will be removed later. In particular, they guarantee that the preference degree of a solution, say $pref(s)$, that we will define later, is a value that could be obtained in the given UBCSP. The importance of considering such constraints is explained in Example 8.

However, the new projection constraints for the the bipolar context are defined in a different way from those in the fuzzy case, since in the bipolar problems there may be negative preferences different from fuzzy preferences and also positive preferences. Nevertheless, it is easy to check that the new approach to define these projection constraints generalizes the fuzzy one.

The set of projection constraints C_{proj} is defined by the function *ComputeProjectionConstraint* in algorithm *B-SP*. Such a function takes in input a bipolar constraint $c = \langle def, con\rangle$ in C_{cu}, such that $con \cap V_c = X$ and $con \cap V_u = Z$, and it returns constraints $\langle defp, X\rangle$ and $\langle defn, X\rangle$, where $defp(t_X) = \inf_{\{t_Z \in A_Z | \pi_Z(a) > 0\}} pos(c)\ (t_X, t_Z)$ and $defn(t_X) = \sup_{\{a \in A_Z\} | \pi_Z(a) > 0} neg(c)\ (t_X, t_Z)$. In other words, $defn(t_X)$ (resp., $defp(t_X)$) is the best negative (resp., the worst positive) preference that could be reached for t_X in c when we consider the various values t_Z in the domain of the uncontrollable variables in Z.

Example 5. Consider the constraint $c_1 = \langle q, \{x, z_1\}\rangle$ in Figure 3 (a), the projection constraints obtained from it are the constraints $p1 = \langle qp, \{x\}\rangle$ and $p2 = \langle qn, \{x\}\rangle$ shown in Figure 3 (b). We recall that in this example positive preferences are in $[0, 1]$ and negative preferences are $[-1, 0]$ and all the preferences are ordered via the maximum operator. In this example, every assignment t_x to the controllable variable x in $p1$ has positive preference equal to 0, since 0 is the worst positive preference associated by $pos(c_1)$ to t_x, and in $p2$ has negative preference equal to 0, since 0 is the best negative preference associated by $neg(c_1)$ to t_x. □

Example 6. Let us consider the UBCSP $Q = \langle S, V_c = \{x, y\}, V_u = \{z_1, z_2\}, \pi = \{p_1, p_2\},\ C_c, C_{cu}\rangle$ in Figure 3 (a). Figure 3 (b) shows the RBCSP $Q' = \langle S, V_c =$

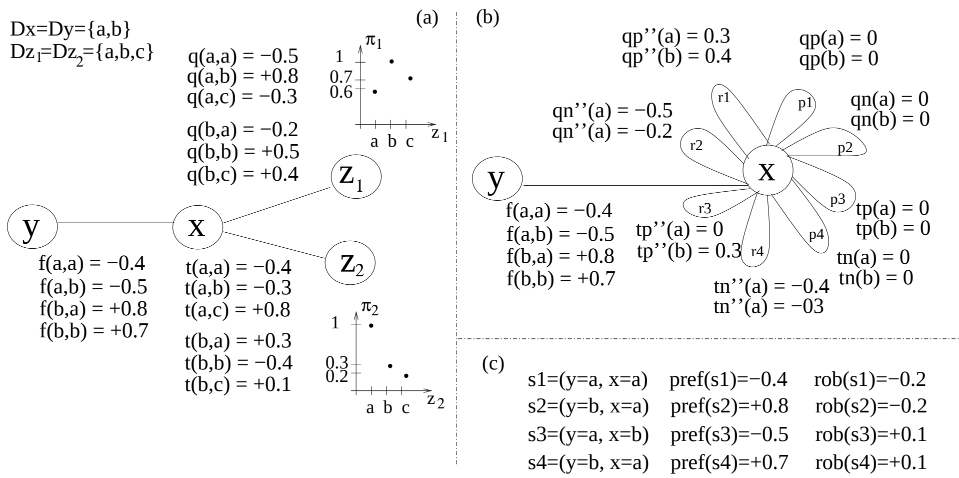

Fig. 3. How *B-SP* works.

$\{x, y\}, C_c^*, C_{rob}\rangle$, where $C_c^* = C_c \cup C_{proj}$, built by algorithm *B-SP*. C_c is composed by $\langle f, \{x, y\}\rangle$. C_{proj} is composed by $\langle qp, \{x\}\rangle$, $\langle qn, \{x\}\rangle$, $\langle tp, \{x\}\rangle$ and $\langle tn, \{x\}\rangle$, while C_{rob} by $\langle qp'', \{x\}\rangle$, $\langle qn'', \{x\}\rangle$, $\langle tp'', \{x\}\rangle$ and $\langle tn'', \{x\}\rangle$. Constraints in C_{rob} are obtained by using functions g_p and g_n as in Example 4. □

6 Preference and robustness

We are now ready to define the preference and the robustness of a solution in an UBCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$. To do that we generalize to the bipolar context the definition of preference and robustness given for the fuzzy case [26]. The main idea is to use algorithm *B-SP* to produce the RBCSP $Q' = \langle S, V_c, C_c^*, C_{rob}\rangle$, where $C_c^* = C_c \cup C_{proj}$, and then to associate to each solution of Q', i.e., to every complete assignment to controllable variables, a pair composed by a degree of preference and a degree of robustness.

Definition 4 (preference). *Given a solution s of an UBCSP Q, let $Q' = \langle S, V_c, C_c^*, C_{rob}\rangle$, where $C_c^* = C_c \cup C_{proj}$, the RBCSP obtained from Q by algorithm* B-SP. *Then the preference of s is*

$$pref(s) = pref_p(s) \times pref_n(s),$$

where

- $\times$ *is the compensation operator of S,*
- $pref_p(s) = \Pi_{\{\langle def, con\rangle \in C_c^*\}}\ pos(c)(s \downarrow_{con})$,
- $pref_n(s) = \Pi_{\{\langle def, con\rangle \in C_c^*\}}\ neg(c)(s \downarrow_{con})$.

In other words, the preference of a solution is obtained by compensating a positive and a negative preference, where the positive (resp., negative) preference is obtained by combining all positive (resp., negative) preferences of the appropriate subtuples of the solution over the constraints in C_c^*, i.e., over the constraints in $C_c \cup C_{proj}$, that are the initial constraints of Q linking only controllable variables and the new projection constraints.

In the following proofs we will sometimes need to use a preference value that we call *projection preference*. More precisely, we will denote the projection preference of a solution s with

$$proj(s) = proj_p(s) \times proj_n(s),$$

where $\times$ is the compensation operator of S, $proj_p(s) = \Pi_{\{\langle def,con\rangle \in C_{proj}\}}\ pos(c)(s \downarrow_{con})$, and $neg(c)(s \downarrow_{con})$.

Definition 5 (robustness). *Given a solution s of an UBCSP Q, let $Q' = \langle S, V_c, C_c^*, C_{rob}\rangle$, where $C_c^* = C_c \cup C_{proj}$, the RBCSP obtained from Q by algorithm* B-SP. *Then the robustness of s is*

$$rob(s) = rob_p(s) \times rob_n(s),$$

where

- $\times$ *is the compensation operator of S,*
- $rob_p(s) = \Pi_{\{\langle def,con\rangle \in C_{rob}\}}\ pos(c)(s \downarrow_{con})$,
- $rob_n(s) = \Pi_{\{\langle def,con\rangle \in C_{rob}\}}\ neg(c)(s \downarrow_{con})$.

In other words, the robustness of a solution is obtained compensating a positive and a negative robustness, where the positive (resp., negative) robustness is obtained by combining all positive (resp., negative) preferences of the appropriate subtuples of the solution over the constraints in C_{rob}, i.e., over robustness constraints.

Notice that, when positive preferences are missing and when the negative preferences are only of the fuzzy kind, the definitions of preference and robustness given above for the bipolar case coincide with those given in [26] for the fuzzy case.

Example 7. Let us consider the UBCSP Q in Figure 3 (a) and the RBCSP Q' obtained from Q by algorithm *B-SP*. Figure 3 (c) shows all the solutions of Q, i.e., all the complete assignments to the controllable variables (thus x and y) with their associated preference and robustness degrees. $\square$

In the following we will show why it is important to add projection constraints. Such constraints avoid having solutions s with the negative preference $pref_n(s)$ better than the best negative preference that could result from C_{cu} constraints and with the positive preference $pref_p(s)$ worse than the worst positive preference that could result from C_{cu} constraints.

Example 8. Consider an UBCSP Q defined over the bipolar preference structure considered before, i.e., $\langle N = [-1, 0], P = [0, 1], + = max, \times, \bot = -1, \square = 0, \top = 1\rangle$, where $\times$ is s.t. $\times_p = max$, $\times_n = min$ and $\times_{np} = sum$. Assume to have a solution s with $pref_n(s) = -0.7$ and $pref_p = +0.5$. Then $pref(s) = pref_n(s) \times_{np} pref_p(s) = -0.7 + 0.5 = -0.2$. Assume also that the best negative preference that can be obtained for s from constraints in C_{cu} is -0.9 and that the worst positive preference that can be obtained for s from constraints in C_{cu} is $+0.9$. Then the best negative preference that can be obtained by s in Q is $pref'_n(s) = pref_n(s) \times_n (-0.9) = min(-0.7, -0.9) = -0.9$, i.e., a negative preference which is strictly worse than $pref_n(s) = -0.7$. Moreover, the worst positive preference that can be obtained for s in Q is $pref'_p(s) = pref_p(s) \times_p (+0.9) = max(+0.5, +0.9) = +0.9$, i.e., a positive preference which is strictly better than $+0.5$. Therefore, the preferences that can be obtained for s in Q are in $[-1, -0.9]$ and in $[0.9, 1]$. Thus, $pref(s) = -0.7 \times_{np} 0.5 = -0.2$ cannot be obtained in Q for s, since in Q the best preference that can obtained for s is $0.1 = -0.9 \times_{np} 1$ and the worst preference that can be obtained for s is $-0.1 = -1 \times_{np} 0.9$. Instead, if we associate to s the preference $pref'(s) = pref'_n(s) \times pref'_p(s) = -0.9 + 0.9 = 0$, then we are sure that such a preference can be really obtained for s in Q. Thus, the addition of projection constraints guarantees that every solution has a preference which can be really obtained in the original problem Q. $\square$

Note that even if we keep separate the positive and negative preferences during algorithm *B-SP*, we compute the preference and robustness of a solution by compensating its positive and its negative components, thus we don't lose information. The only loss of information that we have is due the effect of the possibly non-associative compensation operator. This feature is inherited from BCSPs [8, 10], where associativity is not required in order to allow for a more general framework, that is desirable in practice. However, as said before, in [8, 10] it is also shown a procedure for building a bipolar preference structure with an associative compensation operator.

It is possible to prove that the desired properties on the robustness (i.e., Property BP1 and Property BP2 [25]) presented previously hold.

Theorem 1. *The definition of robustness given in Definition 5 satisfies Property BP1.*

Proof. Consider two solutions, say s and s', of a UBCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu} \rangle$, where $S = \langle N, P +, \times, \bot, \Box, \top \rangle$ is a bipolar preference structure such that P and N are closed intervals of $\mathbb{R}$. For every bipolar constraint $c_i = \langle def_i, con_i \rangle \in C_{cu}$, let us denote with X_i the set $con_i \cap V_c$, with Z_i the set $con_i \cap V_u$, and with π_{Z_i} the possibility distribution associated to Z_i. Assume that, for every such constraint c_i, $\forall t_{Z_i}$ assignment to Z_i, $def_i(s \downarrow_{X_i}, t_{Z_i}) \leq_S def_i(s' \downarrow_{X_i}, t_{Z_i})$, To prove Property BP1, we will show that $rob(s) \leq_S rob(s')$.

Let us denote with t_{X_i} the value $s \downarrow_{X_i}$, with t'_{X_i} the value $s' \downarrow_{X_i}$, and with AZ_i the set of assignments of Z_i. With this notation the hypothesis can be written as follows: $\forall t_{Z_i} \in AZ_i$, $def_i(t_{X_i}, t_{Z_i}) \leq_S def_i(t'_{X_i}, t_{Z_i})$. This holds both for the positive preferences of c_i and for the negative preferences of c_i. In particular, we have that $\forall t_{Z_i} \in AZ_i$, $pos(c_i)(t_{X_i}, t_{Z_i}) \leq_S pos(c_i)(t'_{X_i}, t_{Z_i})$, and $neg(c_i)(t_{X_i}, t_{Z_i}) \leq_S neg(c_i)(t'_{X_i}, t_{Z_i})$. We now consider the case of positive preferences. The case of negative preferences can be dealt similarly.

If, $\forall t_{Z_i} \in AZ_i, pos(c_i)(t_{X_i}, t_{Z_i}) \leq_S pos(c_i)(t'_{X_i}, t_{Z_i})$, then, since the map g_p, that we have defined in Section 5.1, is monotone, $\forall t_{Z_i} \in AZ_i$, $g_p(pos(c_i)(t_{X_i}, t_{Z_i})) \leq_S g_p(pos(c_i)(t'_{X_i}, t_{Z_i}))$. Since the sup operator is monotone, $\forall t_{Z_i} \in AZ_i$, $sup(g_p(pos(c_i)$ $(t_{X_i}, t_{Z_i})), c_S(\pi_{Z_i}(t_{Z_i}))) \leq_S sup(g_p(pos(c_i)(t'_{X_i}, t_{Z_i})), c_S(\pi_{Z_i}\ (t_{Z_i})))$. Moreover, $inf_{t^*_{Z_i} \in AZ_i} sup(g_p(pos(c_i)(t_{X_i}, t^*_{Z_i})), c_S(\pi_{Z_i}(t^*_{Z_i}))) \leq_S sup(g_p(pos(c_i)(t_{X_i}, t_{Z_i})),$

$c_S(\pi_{Z_i}(t_{Z_i})))$, $\forall t_{Z_i} \in AZ_i$. By the previous step, $\forall t_{Z_i} \in AZ_i$, $sup(g_p(pos(c_i)\ (t_{X_i},\ t_{Z_i})),\ c_S(\pi_{Z_i}(t_{Z_i}))) \leq_S sup(g_p(pos(c_i)\ (t'_{X_i},\ t_{Z_i})),\ c_S(\pi_{Z_i}\ (t_{Z_i})))$, thus this holds also for $t^{**}_{Z_i} \in AZ_i$ such that $inf_{t^*_{Z_i} \in AZ_i} sup(g_p(pos(c_i)(t'_{X_i}, t^*_{Z_i})),\ c_S(\pi_{Z_i}\ (t^*_{Z_i}))) = sup(g_p(pos(c_i)\ (t'_{X_i}, t^{**}_{Z_i})), c_S(\pi_{Z_i}\ (t^{**}_{Z_i})))$. Therefore, $inf_{t^*_{Z_i} \in AZ_i}\ sup(g_p(pos(c_i)(t_{X_i},\ t^*_{Z_i})),\ c_S(\pi_{Z_i}(t^*_{Z_i}))) \leq_S inf_{t^*_{Z_i} \in AZ_i}\ sup(g_p(pos(c_i)(t'_{X_i}, t^*_{Z_i})),\ c_S(\pi_{Z_i}\ (t^*_{Z_i})))$. Since the map g_p^{-1}, defined in Section 5.1, is monotone, then the following relation holds: $g_p^{-1}(inf_{t^*_{Z_i} \in AZ_i}\ sup(g_p(pos(c_i)\ (t_{X_i},\ t^*_{Z_i})),\ c_S(\pi_{Z_i}(t^*_{Z_i})))) \leq_S g_p^{-1}(inf_{t^*_{Z_i} \in AZ_i}\ sup\ (g_p(pos(c_i)(t'_{X_i},\ t^*_{Z_i})),\ c_S(\pi_{Z_i}\ (t^*_{Z_i}))))$, i.e., with the notation used in Section 5.1 for defining one the robustness constraint in C_{rob} corresponding to $c_i \in C_{cu}$, $defp''_i(t_{X_i}) \leq_S defp''_i(t'_{X_i})$. By monotonicity of $\times_p$, if we combine via $\times_p$ all such constraints $\langle defp''_i,\ X\rangle$ we have that, $\prod_{p\ \langle defp''_i, X_i\rangle} defp''_i(t_{X_i}) \leq_S \prod_{p\ \langle defp''_i, X_i\rangle} defp''_i(t'_{X_i})$.

Similarly, using the same notation presented in Section 5.1, it can be shown that $\prod_{\langle defn''_i, X_i\rangle} defn''_i(t_{X_i}) \leq_S \prod_{\langle defn''_i, X_i\rangle} defn''_i(t'_{X_i})$, since the maps g_n and g_n^{-1}, described in Section 5.1, are monotone and since the $\times_n$ operator is monotone.

By definition 5, $rob(s) = rob_p(s) \times rob_n(s)$, where $rob_p(s) = \prod_{p\ c=\langle def,con\rangle \in C_{rob}} pos(c)(s \downarrow con)$ and $rob_n(s) = \prod_{n\ c=\langle def,con\rangle \in C_{rob}} neg(c)(s \downarrow con)$. Since $rob_p(s) = \prod_{p\ c=\langle def,con\rangle \in C_{rob}} pos(c)(s \downarrow con) = \prod_{\langle defp''_i, X_i\rangle} defp''_i(t_{X_i})$, and since $rob_n(s) = \prod_{n\ c=\langle def,con\rangle \in C_{rob}} neg(c)(s \downarrow con) = \prod_{\langle defp''_i, X_i\rangle} defp''_i(t'_{X_i})$, we can conclude, by the previous step, that $rob_p(s) \leq_S rob_p(s')$, $rob_n(s) \leq_S rob_n(s')$, and thus, since the $\times$ operator is monotone, that $rob(s) \leq_S rob(s')$. □

Theorem 2. *The definition of robustness given in Definition 5 satisfies Property BP2.*

Proof. Consider a solution s of the UBCSPs $Q_1 = \langle S, V_c, V_u, \pi_1, C_c,\ C_{cu}\rangle$ and $Q_2 = \langle S, V_c, V_u, \pi_2, C_c,\ C_{cu}\rangle$, where $S = \langle N,\ P\ +,\ \times,\ \bot,\ \Box,\ \top\rangle$ is a bipolar preference structure such that P and N are closed intervals of $\mathbb{R}$ Assume that for every assignment t_Z to the uncontrollable variables in V_u, $\pi_2(t_Z) \leq \pi_1(t_Z)$. To prove Property BP2, we will show that $rob_{\pi_1}(s) \leq_S rob_{\pi_2}(s)$, where rob_{π_1} is the robustness computed in the problem with possibility distribution π_1, and rob_{π_2} is the robustness computed in the problem with possibility distribution π_2.

Assume the notation considered in the first part of the proof of Theorem 1. By hypothesis, we know that $\forall t_{Z_i} \in AZ_i$, $\pi_2(t_{Z_i}) \leq \pi_1 t_{Z_i}$. Since c_S is an order reversing map, $\forall t_{Z_i} \in AZ_i$, $c_S(\pi_2(t_{Z_i})) \geq_S c_S(\pi_1 t_{Z_i})$. By monotonicity of the sup operator, $\forall t_{Z_i} \in AZ_i$, $sup(g_p(pos(c_i)\ (t_{X_i}, t_{Z_i})), c_S(\pi_1(t_{Z_i}))) \leq_S sup(g_p(pos(c_i)(t_{X_i}, t_{Z_i})),\ c_S(\pi_2\ (t_{Z_i})))$ and $sup(g_p(neg(c_i)\ (t_{X_i}, t_{Z_i})), c_S(\pi_1(t_{Z_i}))) \leq_S sup(g_p(neg(c_i)(t_{X_i},\ t_{Z_i})),\ c_S(\pi_2\ (t_{Z_i})))$. From here we can conclude as in the proof of Theorem 1. □

The proofs of the Theorems 1 and 2 are based on the fact that the preference functions in the robustness constraints, which are used to build rob_p and rob_n of a solution, are obtained by using functions g_p and g_n (mapping resp. positive and negative preferences in $[0, 1]$) which are strictly monotonic, and on the fact that the operators $\times_p$, used for computing rob_p of a solution, $\times_n$, used for computing rob_n of a solution, and $\times$, used for compute rob of a solution, are monotonic. The proof regarding Property BP2 depends also on the fact that c_S is an order reversing map w.r.t. $\leq_S$, and thus if $\pi_1(a) \leq \pi_2(a)$, then $c_S(\pi_1(a)) \geq_S c_S(\pi_2(a))$.

7 Semantics

A solution of a BCSP is associated to a preference and a robustness degree as in the fuzzy approach [26]. In Section 2.7 we have recalled some of the most significative semantics (i.e., Risky, Safe, and Diplomatic) used in [26] to order the solutions which depend on our attitude w.r.t. preference and robustness. We now generalize these semantics to the bipolar context as follows.

More precisely, let $\leq_S$ the ordering induced by the additive operator of the bipolar preference structure of the considered UBCSP (and not the ordering induced by the additive operator of the c-semiring of considered USCSP as in [26]),

- **Risky** semantics is a lexicographic ordering w.r.t. $\leq_S$ on pairs $\langle pref, rob \rangle$, that gives more importance to the preference degree: given $A1 = (pref_1, rob_1)$ and $A2 = (pref_2, rob_2)$, $A1 \succ_{Risky} A2$ iff $pref_1 >_S pref_2$ or ($pref_1 = pref_2$ and $rob_1 >_S rob_2$). It gives more relevance to the preference that can be reached in the best case considering less important a high risk of being inconsistent.
- **Safe** semantics is a lexicographic ordering w.r.t. $\leq_S$ on pairs $\langle pref, rob \rangle$, that gives more importance to the robustness degree: given $A1 = (pref_1, rob_1)$ and $A2 = (pref_2, rob_2)$, $A1 \succ_{Safe} A2$ iff $rob_1 >_S rob_2$ or ($rob_1 =_S rob_2$ and $pref_1 >_S pref_2$).
- **Diplomatic** semantics aims at giving the same importance to preference and robustness. It is a Pareto ordering w.r.t. $\leq_S$ (and not w.r.t. $\leq$ as in the fuzzy case) on pairs $\langle pref, rob \rangle$: given $A1 = (pref_1, rob_1)$ and $A2 = (pref_2, rob_2)$, $A1 \succ_{Dipl} A2$ iff ($pref_1 \geq_S pref_2$ and $rob_1 \geq_S rob_2$) and ($pref_1 >_S pref_2$ or $rob_1 >_S rob_2$).

Example 9. Let us consider the UBCSP Q in Figure 3 (a). In Figure 3 (c) all the solutions of Q are shown with their associated preference and robustness degrees. The optimal solution for the Risky semantics is $s_2 = (y = b, x = a)$, which has preference 0.8 and robustness $-0, 2$, while for the Safe semantics is $s_4 = (y = b, x = b)$, which has preference 0.7 and robustness 0.1. For the Diplomatic semantics, s_2 and s_4 are equally optimal. Note that the solutions chosen by the various semantics differ on the attitude toward risk they implement. In fact, Risky chooses the solution that gives a high positive preference in the controllable part, even if the uncontrollable part has a high possibility of a negative preference. On the other hand, for the Safe semantics it is better to select a solution with a higher robustness, i.e., that guarantees a higher number of scenarios with a higher preference. In this example, Safe chooses a solution with a lower preference with respect to Risky, but that will have with high possibility a positive preference in the part involving uncontrollable variables. □

By definition of Risky, Safe and Diplomatic semantics, it follows that for these semantics the desired properties on solution ordering (i.e., Properties BP3 and BP4) presented previously hold.

Theorem 3. *The solution orderings $\succ_{Risky}$, $\succ_{Safe}$ and $\succ_{Diplomatic}$ satisfy Property BP3.*

Proof. Property BP3 states that, given two solutions s and s' of an UBCSP, if $rob(s) = rob(s')$ and $pref(s) >_S pref(s')$, $s \succ s'$. By definition of Risky, Safe and Diplomatic semantics, this property holds for $\succ_{Risky}$, $\succ_{Safe}$ and $\succ_{Dipl}$. □

Theorem 4. *The solution orderings $\succ_{Risky}$, $\succ_{Safe}$ and $\succ_{Diplomatic}$ satisfy Property BP4.*

Proof. Property BP4 states that, given two solutions s and s' of an UBCSP, if $pref(s) = pref(s')$ and $rob(s) >_S rob(s')$, $s \succ s'$. By definition of Risky, Safe and Diplomatic semantics, this property holds for $\succ_{Risky}$, $\succ_{Safe}$ and $\succ_{Dipl}$ $\square$.

Also, it is possible to prove that Property BP5 is satisfied only by $\succ_{Risky}$.

Theorem 5. *Given an UBCSP $\langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$, the solution ordering $\succ_{Risky}$ satisfies Property BP5 if the operator $\times$ of S is strictly monotonic, while the solution orderings $\succ_{Safe}$ and $\succ_{Diplomatic}$ never satisfy Property BP5.*

Proof. To prove Property BP5, we have to show that, given two solutions s and s' of a UBCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$, such that $ovpref_p(s, a) >_S ovpref_p(s', a)$ and $ovpref_p(s, a) >_S ovpref_p(s', a)$ $\forall a$ assignment to V_u, then $s \succ_{Risky} s'$.

From UBCSP Q we can obtain an equivalent problem that corresponds to the UBCSP $QP = \langle S, \{V^c\}, \{V^u\}, C_{1p} \cup C_{1n} \cup C_{3p} \cup C_{3n}, C_{2p} \cup C_{2n}\rangle$, where we recall separately the sets of constraints $C_{1p}, C_{1n}, C_{3p}, C_{3n}$, C_{2p}, and C_{2n}. In QP the element V^c is a controllable variable and V^u is an uncontrollable variable, representing respectively all the variables in V_c and V_u, having as domains the corresponding Cartesian products. The uncontrollable variable V^u is described by a possibility distribution, π, which is the joint possibility, i.e., the possibility obtained by performing the minimum among all the possibility distributions of the uncontrollable variables in V_u. Constraint $C_{1p} = \langle defp_1, V^c\rangle$ (resp., $C_{1n} = \langle defn_1, V^c\rangle$) is defined as the combination of all constraints in C_c connecting variables in V_c, where the negative (resp., positive) preferences are interpreted as indifference. Constraint $C_{2p} = \langle defp_2, \{V^c, V^u\}\rangle$ (resp., $C_{2n} = \langle defn_2, \{V^c, V^u\}\rangle$) is the combination of all the constraints in C_{cu} connecting variables in V_c to variables in V_u, where the negative (resp., positive) preferences are interpreted as indifference. Constraint $C_{3p} = \langle defp_3, V^c\rangle$ (resp., $C_{3n} = \langle defn_3, V^c\rangle$) is defined as the combination of all the constraints obtained from constraints in C_2, interpreting the negative (resp., positive) preferences as indifference, and by projecting them over the controllable variables in V_c as described in Section 5.2. Notice that all these combinations are obtained using operator $\times_p$ (resp., $\times_n$) of the c-semiring S. Thus, given an assignment s to V^c in Q, which corresponds to an assignment to all the variables in V_c, its preference on constraint C_{1p} is $defp_1(s) = \prod_{c_i=\langle def_i, con_i\rangle \in C_c} pos(c_i)(s \downarrow con_i) = control_p(s)$, on C_{3p} is $defp_3(s) = proj_p(s)$, and on $C_1 \bigotimes C_3$ is $defp_1(s) \times defp_3(s) = control_p(s) \times proj_p(s) = pref_p(s)$. Given assignment (s, a_i) to (V^c, V^u), instead, which corresponds to a complete assignment to variables in V_c and V_u, its preference, $defp_2(s, a_i)$ (resp., $defn_2(s, a_i)$), is obtained by performing the combination of the positive (resp. negative) preferences associated to all the subtuples of (s, a_i) by the constraints in C_{cu}, interpreting the negative (resp., positive) preferences as indifference. Using this new notation we have that, $\forall (s, a_i)$ assignments to V^c and V^u, $ovpref_p(s, a_i) = defp_1(s) \times defp_2(s, a_i) = control_p(s) \times defp_2(s, a_i)$, and $ovpref_n(s, a_i) = defn_1(s) \times defn_2(s, a_i) = control_n(s) \times defn_2(s, a_i)$.

If we show that $pref_p(s, a_i) >_S pref_p(s, a_i)$ and $pref_n(s, a_i) >_S pref_n(s, a_i)$, $\forall a_i$ assignment to V^u, then, by strict monotonicity of the $\times$ operator, we can conclude that $pref(s) = pref_p(s) \times pref_n(s) >_S pref_p(s') \times pref_n(s') = pref(s')$, and thus that $s \succ_{Risky} s'$.

We first show that $pref_p(s) >_S pref_p(s')$. We know, by hypothesis, that $ovpref_p(s, a_i) >_S ovpref_p(s', a_i)$, $\forall a_i$ assignment to V^u, i.e., that $control_p(s) \times defp_2(s, a_i) >_S control_p(s') \times defp_2(s', a_i)$, $\forall a_i$ assignment to V^u. This must hold also for the assignment to V^u, that we call a^*, such that $defp_2(s, a^*) = proj_p(s)$. Hence, $pref_p(s) = control_p(s) \times proj_p(s) = control_p(s) \times defp_2(s, a^*) >_S control_p(s') \times defp_2(s', a^*)$. Moreover, since, by definition of $proj_p$ (see Sections 5.2 and 6), $proj_p(s') \leq_S defp_2(s', a_i), \forall a_i$, we have that $control_p(s') \times defp_2(s', a^*) \geq_S control_p(s') \times proj_p(s') = pref_p(s')$, and thus $pref_p(s) >_S pref_p(s')$.

To conclude that $s \succ_{Risky} s'$, we have to show that $pref_n(s) >_S pref_n(s')$. We know, by hypothesis, that $ovpref_n(s, a_i) >_S ovpref_n(s', a_i)$, $\forall a_i$ assignment to V^u, i.e., that $control_n(s) \times defn_2(s, a_i) >_S control_n(s') \times defn_2(s', a_i)$, $\forall a_i$ assignment to V^u. This must hold also for the assignment to V^u, that we call a^*, such that $defn_2(s', a^*) = proj_n(s')$. Hence, $control_n(s) \times def_2(s, a^*) >_S control_n(s') \times proj_n(s') = pref_n(s')$. Moreover, since by definition of the $proj_n$ (see Sections 5.2 and 6), $proj_n(s) \geq_S defn_2(s, a_i), \forall a_i$, we have that $pref_n(s) = control_n(s) \times proj_n(s) \geq_S control_n(s) \times defn_2(s, a^*) >_S pref_n(s')$, and thus $pref_n(s) >_S pref_n(s')$.

We now show that Property BP5 is not satisfied by $\succ_{Safe}$ and $\succ_{Dipl}$. For these semantics it can happen that $s \not\succ s'$. In fact, let us consider the UBCSP $Q = \langle S_{FCSP}, V_c, \pi, V_u, C_c, C_{cu} \rangle$, where the bipolar preference structure is the fuzzy c-semiring $\langle [0, 1], max, min, 0, 1 \rangle$, $V_c = \{x\}$, $V_u = \{z\}$, C_c is composed by $c_1 = \langle f_1, \{x\} \rangle$, C_{cu} by $c_2 = \langle f_2, \{x, z\} \rangle$, and where $D_z = \{a_1, a_2\}$ and $D_x = \{s, s'\}$ are respectively the domain of z and x. Let us assume that the possibility distribution on z is such that $\pi(a_1) = 1$ and $\pi(a_2) = 0.7$. Let us assume moreover that $f_2(s, a_1) = 0.4$, $f_2(s, a_2) = 0.5$, $f_2(s', a_1) = 0.8$, $f_2(s', a_2) = 0.9$, $f_1(s) = 0.3$ and $f_1(s') = 0.2$. The overall preferences are: $ovpref(s, a_1) = 0.3$, $ovpref(s, a_2) = 0.3$, $ovpref(s', a_1) = 0.2$, $ovpref(s', a_2) = 0.2$, i.e., $ovpref(s, a_i) > ovpref(s', a_i), \forall a_i$, $i = 1, 2$, hence s and s' satisfy the hypothesis. The robustness values for s and s' (computed considering as g_n the identity map) are $rob(s) = inf(max(0.4, 0), max(0.5, 0.3)) = 0.4$, $rob(s) = inf(max(0.8, 0), max(0.9, 0.3)) = 0.8$. Therefore, since $rob(s) < rob(s')$, $s \prec_{Safe} s'$ for Safe semantics. The preference degrees are $pref(s) = min(control(s), proj(s)) = min(0.3, 0.5) = 0.3$ and $pref(s') = min(control(s'), proj(s')) = min(0.2, 0.9) = 0.2$. Since $rob(s) < rob(s')$ and $pref(s) > pref(s')$, $s \bowtie_{Dipl} s'$ for Diplomatic semantics. □

We have shown before that Risky, Safe and Diplomatics semantics for UBCSPs satisfy Property BP3 and BP4 and that Risky satisfies also Property BP5. However, there are semantics that don't satisfy them. Consider for example a semantics, that we call **Mixed**, such that given $A1 = (pref_1, rob_1)$ and $A2 = (pref_2, rob_2)$, $A1 \succ_{Mixed} A2$ iff $pref_1 \times rob_1 >_S pref_2 \times rob_2$, where $\times$ is the compensation operator in the considered bipolar preference structure. This semantics generalizes the one adopted to order the solutions in [17] for fuzzy c-semiring $\langle [0,1], max, min, 0, 1\rangle$. It is possible to show that Mixed semantics does not satisfy properties BP3, BP4 and BP5.

8 Extending the approach to UBCSPs with totally ordered positive/negative preferences

In the previous sections we have shown a procedure for handling UBCSPs where the set of the positive preferences (P) and the set of the negative preferences (N) are two closed intervals of $\mathbb{R}$ (for example, $P = [3,5]$ and $N = [-3,-2]$). In this section we will show that it is possible to generalize this method to more general bipolar problems where the set of the positive preferences and the set of the negative preferences are totally ordered sets that are not necessarily closed intervals of $\mathbb{R}$. For example,

- they can be real intervals including $+\infty$ or $-\infty$ (for example, $P = [5, +\infty]$ and $N = [-\infty, -8]$),
- they can be the union of disjoint intervals of $\mathbb{R} \cup \{+\infty, -\infty\}$ (for example, $P = [1,3] \cup [5,+\infty]$ and $N = [-\infty, -8] \cup [-3,-2]$)),
- they can be generic totally ordered sets (for example, $P = \{a, b, c\}$ where $a > b > c$ and $N = \{d, e, f\}$ where $d > e > f$).

To show that the new approach generalizes the previous one, we we will show that the same desirable properties continue to hold.

We recall that the main idea to handle UBCSPs over closed real intervals is to remove uncertainty from them, recalling as much information as possible. In particular, the adopted procedure (see Section 5) takes as input a UBCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$, with $S = \langle N, P, +, \times, \bot, \Box, \top\rangle$, where $P = [a_p, b_p]$ and $N = [a_n, b_n]$ are two closed intervals of $\mathbb{R}$, i.e.., two intervals of $\mathbb{R} - \{-\infty, +\infty\}$, it removes uncertainty from Q, by eliminating the uncontrollable variables and all the constraints in C_{cu} relating controllable and uncontrollable variables, and by adding new constraints, i.e., C_{proj} and C_{rob}, only among these controllable variables.

The part of such a procedure that requires that positive and negative preferences are two intervals of $\mathbb{R} - \{-\infty, +\infty\}$ is the one regarding the addition of constraints in C_{rob} (see Section 5.1). We recall that it works as follows. In the first step it translates every positive (resp., negative) preference of the constraints in C_{cu} in $[0,1]$, via the map g_p: $[a_p, b_p] \to [0,1]$ such that $x \mapsto \frac{x-a_p}{b_p-a_p}$, (resp., g_n: $[a_n, b_n] \to [0,1]$ such that $x \mapsto \frac{x-a_n}{b_n-a_n}$), to be able to compare, in the second step, preferences and possibilities, since the possibilities are defined in $[0,1]$. Then, in the third step, it translates the preferences in $[0,1]$ obtained so far in P (resp., N), i.e., in the set of positive (resp., negative) preferences defined in S, by using the inverse map g_p^{-1}:$[0,1] \to [a_p, b_p]$ such that $y \mapsto [y(b_p - a_p) + a_p]$, (resp., g_n^{-1}:$[0,1] \to [a_n, b_n]$ such that $y \mapsto [y(b_n - a_n) + a_n]$.

The functions g_p, g_n, g_p^{-1}, and g_n^{-1} mentioned above have been used to prove that some of the desirable properties hold (see proofs of Proposition 2, Theorem 1, and Theorem 2). In these proofs, for what concerning the functions above, we have only used the fact that g_p and g_p^{-1} (resp., g_n and g_n^{-1}) are monotonic, and that their combinations gives the identity map.

To extend the approach to UBCSPs where the sets of positive and negative preferences are generic totally ordered sets, we can use, instead of g_p and g_p^{-1} (resp., g_n and g_n^{-1}), two functions that define a Galois insertion (see Section 8.1), since in this case we are sure that they are both monotonic, and their combination is the identity map.

8.1 Galois insertions

In this section we give the notion of *Galois insertions*, that we will consider in our generalized procedure, and we insert such a definition in the context of abstract interpretation [5].

Abstract interpretation [4, 12] is a theory developed to reason about the relation between two different semantics (the *concrete* and the *abstract* semantics). The idea of approximating program properties by evaluating a program on simpler domain of descriptions of "concrete" program states goes back to the early 70's. The guiding idea is to relate the concrete and the abstract interpretations of the calculus by a pair of functions, the *abstraction function* α and the *concretization* function γ, which form a Galois connection.

Let ($\mathcal{C}$, $\leq$) (concrete domain) be the domain of the concrete semantics, while ($\mathcal{A}$, $\sqsubseteq$) (abstract domain) be the domain of the abstract semantics. The partial order relations reflect an approximation relation. Since in approximation theory a partial order specifies the precision degree of any element in a poset, it is obvious assume that if α is a mapping associating an abstract object in ($\mathcal{A}$, $\sqsubseteq$) for every concrete element in ($\mathcal{C}$, $\leq$), then the following holds: if $\alpha(x) \sqsubseteq y$, then y is also a correct, although less precise, abstract approximation of x. The same argument holds if $x \leq \gamma(y)$. Then y is also a correct approximation of x, although x provides more accurate information than $\gamma(y)$. This gives rise to the following formal definition [5].

Definition 6 (Galois insertion). *Let ($\mathcal{C}$, $\leq$) and ($\mathcal{A}$, $\sqsubseteq$) be two posets (the concrete and the abstract domain). A Galois connection $\langle \alpha, \gamma \rangle$: ($\mathcal{C}$, $\leq$) $\rightleftharpoons$ ($\mathcal{A}$, $\sqsubseteq$) is a pair of maps $\alpha : \mathcal{C} \to \mathcal{A}$ and $\gamma : \mathcal{A} \to \mathcal{C}$ such that*

1. *α and γ are monotonic;*
2. *for each $x \in \mathcal{C}$, $x \leq \gamma(\alpha(x))$, and*
3. *for each $y \in \mathcal{A}$, $\alpha(\gamma(x)) \sqsubseteq y$.*

Moreover, a Galois insertion (of $\mathcal{A}$ and $\mathcal{C}$) $\langle \alpha, \gamma \rangle$: ($\mathcal{C}$, $\leq$) $\rightleftharpoons$ ($\mathcal{A}$, $\sqsubseteq$) is a Galois connection where $\gamma \cdot \alpha = id_{\mathcal{A}}$.

8.2 A generalized approach to UBCSPs with totally ordered preferences

We now show how Galois insertions allow us to extend to UBCSPs over totally ordered sets of positive and negative preferences the procedure described in Section 5.1 to remove uncertainty guaranteeing that the same desired properties continue to hold.

Consider an UBCSP with bipolar preference structure $S = \langle N, P, +, \times, \perp, \Box, \top \rangle$, where P and N are totally ordered sets. Let us denote with $\leq_S$ the ordering induced by the additive operator. Consider also the totally ordered set $[0, 1]$ with the ordering $\sqsubseteq$ such that where $0 \sqsubseteq 1$.

We now redefine the functions g_p and g_p^{-1} presented in Section 5.1 as follows: $\langle g_p, g_p^{-1}\rangle : (P, \leq_S) \rightleftharpoons ([0,1], \sqsubseteq)$ is a Galois insertion. We know, by definition of Galois insertion, that

- $g_p : P \rightarrow [0,1]$ is monotonic, i.e., $\forall x_1, x_2 \in P$, with $x_1 \leq x_2$, $g_p(x_1) \sqsubseteq g_p(x_2)$;
- $g_p^{-1} : [0,1] \rightarrow P$ is monotonic, i.e., $\forall y_1, y_2 \in [0,1]$, with $y_1 \sqsubseteq y_2$, $g_p^{-1}(y_1) \sqsubseteq \gamma(y_2)$;
- $g_p^{-1} \cdot g_p = id$.

Similarly, we redefine the functions g_n and g_n^{-1} presented in Section 5.1 as follows $\langle g_n, g_n^{-1}\rangle : (N, \leq) \rightleftharpoons ([0,1], \sqsubseteq)$ is a Galois insertion.

Note that g_p and g_p^{-1} can be defined in several different ways, but all of them have to satisfy the properties of the Galois insertions, from which it derives, among others, that $g_p(\bot_P) = 0$ and $g_p(\top_P) = 1$, i.e., that the bottom of P must be mapped in 0 and that the top of P must be mapped in 1. The same must hold for g_n and g_n^{-1}.

Moreover, we redefine the map c_S as follows: it is an order reversing map such that $\forall a, b \in [0,1]$, if $a \leq b$, then $c_S(a) \sqsubseteq c_S(b)$, and $\forall p \in [0,1]$, $c_S(c_S(p)) = p$.

It is possible to show that, using the new definitions of g_p, g_p^{-1}, g_n, g_n^{-1}, and c_S, that all the desired properties that have been shown by exploiting these functions (i.e., Proposition 2, Theorem 1, and Theorem 2) continue to hold.

Proposition 3. *Consider an UBCSP $\langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$, where $S = \langle N, P, +, \times, \bot, \Box, \top\rangle$ is a bipolar preference structure where P and N are* totally ordered *sets. For every constraint $c = \langle def, con\rangle \in C_{cu}$ such that $con \cap V_u = Z$, with possibility distribution π_Z, and $con \cap V_u = X$, the corresponding robustness constraints $\langle defp'', X\rangle$ and $\langle defn'', X\rangle$ are such that, for every t_X assignment to X,*

- *$defp''(t_X) \geq_S \beta \in P$ iff, when $\pi_Z(t_Z) > c_S(g_p(\beta))$, then $pos(c)(t_X, t_Z) \geq_S \beta$,*
- *$defn''(t_X) \geq_S \alpha \in N$ iff, when $\pi_Z(t_Z) > c_S(g_n(\alpha))$, then $pos(c)(t_X, t_Z) \geq_S \alpha$,*

where t_Z is an assignment to Z, $\langle g_p, g_p^{-1}\rangle : (P, \leq_S) \rightleftharpoons ([0,1], \sqsubseteq)$ and $\langle g_n, g_n^{-1}\rangle : (N, \leq_S) \rightleftharpoons ([0,1], \sqsubseteq)$ are Galois insertions, and c_S is an order reversing map such that $\forall a, b \in [0,1]$, if $a \leq b$, then $c_S(a) \sqsupseteq c_S(b)$, and $\forall p \in [0,1]$, $c_S(c_S(p)) = p$.

Proof. We show the first statement concerning $defp''(t_X)$. The second one, concerning $defn''(t_X)$, can be proved analogously, since by construction g_n and g_n^{-1} have the same properties respectively of g_p and g_p^{-1}. We recall that $defp''(t_X) = g_p^{-1}(inf_{t_Z \in A_Z} (g_p(pos(c)(t_X, t_Z)) + c_S(\pi_Z(t_Z))))$, where A_Z is the set of the assignment to Z.
($\Rightarrow$) We assume that $defp''(t_X) \geq_S \beta$. If this holds, then, since g_p is monotone, $g_p(defp''(t_X)) \sqsupseteq g_p(\beta)$, i.e., $g_p(g_p^{-1}(inf_{t_Z \in A_Z}\, sup(g_p(pos(c)(t_X, t_Z)), c_S(\pi_Z(t_Z)))))$ $\sqsupseteq g_p(\beta)$, that is, since the combination of g_p and g_p^{-1} produce the identity map, $inf_{t_Z \in A_Z}$ $sup(g_p(pos(c)(t_X, t_Z),\ c_S(\pi_Z(t_Z)))) \sqsupseteq g_p(\beta)$. Since we are considering totally ordered preferences, this implies that $sup(g_p(pos(c)(t_X, t_Z)),\ c_S(\pi_Z(t_Z))) \sqsupseteq g_p(\beta)$, $\forall t_Z \in A_Z$. For t_Z with $\pi_Z(t_Z) > c_S(g_p(\beta))$, by definition of c_S, we have $c_S(\pi_Z(t_Z))$ $\sqsubset c_S(c_S(g_p(\beta)) = g_p(\beta)$. Therefore for such a value t_Z we have that $g_p(pos(c)(t_X, t_Z)) = sup(g_p(pos(c)(t_X, t_Z)), c_S(\pi_Z(t_Z))) \sqsupseteq_S g_p(\beta)$ and, since g_p^{-1} is monotone, we have $g_p^{-1}(g_p(pos(c)(t_X, t_Z))) \geq_S g_p^{-1}(g_p(\beta))$, i.e., $pos(c)(t_X, t_Z) \geq_S \beta$.
($\Leftarrow$) We assume that $\forall t_Z$ with $\pi_Z(t_Z) > c_S(g_p(\beta))$, $pos(c)(t_X, t_Z) \geq_S \beta$. Then, for

such t_Z, since g_p is monotone, $g_p(pos(c)(t_X, t_Z)) \sqsupseteq g_p(\beta)$ and so, $sup(g_p(pos(c)(t_X, t_Z)),\ c_S(\pi_Z(t_Z))) \sqsupseteq\ g_p(\beta)$. On the other hand, for every t_Z such that $\pi_Z(t_Z) < c_S(g_p(\beta))$, we have, by definition of c_S, $c_S(\pi_Z(t_Z)) \sqsupset g_p(\beta)$ and so $sup(g_p(pos(c)(t_X, t_Z)), c_S(\pi_Z(t_Z))) \sqsupset g_p(\beta)$. Thus $\forall t_Z \in A_Z$, $sup(g_p(pos(c)(t_X, t_Z)), c_S(\pi_Z(t_Z))) \sqsupseteq g_p(\beta)$ and so $inf_{t_Z \in A_Z} sup(pos(c)(t_X, t_Z), c_S(\pi_Z(t_Z))) \sqsupseteq g_p(\beta)$. Hence, since g_p^{-1} is monotone, $g_p^{-1}(inf_{t_Z \in A_Z}\ (sup(pos(c)(t_X, t_Z),\ c_S(\pi_Z(t_Z))))) \geq_S g_p^{-1}(g_p(\beta))$, i.e., $defp''(t_X) \geq_S \beta$. $\square$

Consider an UBCSP $\langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$, where $S = \langle N, P, +, \times, \bot, \Box, \top\rangle$ is a bipolar preference structure where P and N are *totally ordered* sets. It is possible to prove that, if we determine the robustness constraints with the new maps g_p, g_p^{-1}, g_n, g_n^{-1}, and c_S defined in this section, the definition of robustness given in Definition 5 satisfies Properties BP1 and BP2.

Theorem 6. *If we determine the robustness constraints described in Section 5.1 with the maps g_p, g_p^{-1}, g_n, g_n^{-1}, and c_S such that $\langle g_p, g_p^{-1}\rangle : (P, \leq_S) \rightleftharpoons ([0,1], \sqsubseteq)$ and $\langle g_n, g_n^{-1}\rangle : (N, \leq_S) \rightleftharpoons ([0,1], \sqsubseteq)$ are Galois insertions, and c_S is an order reversing map such that $\forall a, b \in [0,1]$, if $a \leq b$, then $c_S(a) \sqsupseteq c_S(b)$, and $\forall p \in [0,1]$, $c_S(c_S(p)) = p$, the definition of robustness given in Definition 5 satisfies Property BP1.*

Proof. The first part of proof coincides with the one of Theorem 1.

Consider two solutions, say s and s', of a UBCSP $Q = \langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$, where $S = \langle N, P +, \times, \bot, \Box, \top\rangle$ is a bipolar preference structure such that P and N are totally ordered sets. For every bipolar constraint $c_i = \langle def_i, con_i\rangle \in C_{cu}$, let us denote with X_i the set $con_i \cap V_c$, with Z_i the set $con_i \cap V_u$, and with π_{Z_i} the possibility distribution associated to Z_i. Assume that, for every such constraint c_i, $\forall t_{Z_i}$ assignment to Z_i, $def_i(s \downarrow_{X_i}, t_{Z_i}) \leq_S def_i(s' \downarrow_{X_i}, t_{Z_i})$, To prove Property BP1, we will show that $rob(s) \leq_S rob(s')$. Let us denote with t_{X_i} the value $s \downarrow_{X_i}$, with t'_{X_i} the value $s' \downarrow_{X_i}$, and with AZ_i the set of assignments of Z_i. With this notation the hypothesis can be written as follows: $\forall t_{Z_i} \in AZ_i$, $def_i(t_{X_i}, t_{Z_i}) \leq_S def_i(t'_{X_i}, t_{Z_i})$. This holds both for the positive preferences of c_i and for the negative preferences of c_i. In particular, we have that $\forall t_{Z_i} \in AZ_i$, $pos(c_i)(t_{X_i}, t_{Z_i}) \leq_S pos(c_i)(t'_{X_i}, t_{Z_i})$, and $neg(c_i)(t_{X_i}, t_{Z_i}) \leq_S neg(c_i)(t'_{X_i}, t_{Z_i})$. We now consider the case of positive preferences. The case of negative preferences can be dealt similarly.

The new part of the proof starts from here. If, $\forall t_{Z_i} \in AZ_i$, $pos(c_i)(t_{X_i}, t_{Z_i}) \leq_S pos(c_i)(t'_{X_i}, t_{Z_i})$, then, since the map g_p is monotone, $\forall t_{Z_i} \in AZ_i$, $g_p(pos(c_i)(t_{X_i}, t_{Z_i})) \sqsubseteq g_p(pos(c_i)(t'_{X_i}, t_{Z_i}))$. Since the sup operator is monotone, $\forall t_{Z_i} \in AZ_i$, $sup(g_p(pos(c_i)(t_{X_i}, t_{Z_i})), c_S(\pi_{Z_i}(t_{Z_i}))) \sqsubseteq sup(g_p(pos(c_i)(t'_{X_i}, t_{Z_i})), c_S(\pi_{Z_i}\ (t_{Z_i})))$. Moreover, we have $inf_{t^*_{Z_i} \in AZ_i}\ sup(g_p(pos(c_i)(t_{X_i}, t^*_{Z_i})), c_S(\pi_{Z_i}(t^*_{Z_i}))) \sqsubseteq sup(g_p(pos(c_i)(t_{X_i}, t_{Z_i})), c_S(\pi_{Z_i}(t_{Z_i})))$, $\forall t_{Z_i} \in AZ_i$. By the previous step, $\forall t_{Z_i} \in AZ_i$, $sup(g_p(pos(c_i)\ (t_{X_i}, t_{Z_i})), c_S(\pi_{Z_i}(t_{Z_i}))) \sqsubseteq sup(g_p(pos(c_i)\ (t'_{X_i}, t_{Z_i})),\ c_S(\pi_{Z_i}\ (t_{Z_i})))$, thus this holds also for $t^{**}_{Z_i} \in AZ_i$ such that $inf_{t^*_{Z_i} \in AZ_i} sup(g_p(pos(c_i)(t'_{X_i}, t^*_{Z_i})),\ c_S(\pi_{Z_i}\ (t^*_{Z_i})))$ is equal to $sup(g_p(pos(c_i)(t'_{X_i}, t^{**}_{Z_i})), c_S(\pi_{Z_i}\ (t^{**}_{Z_i})))$. Therefore, we can conclude that $inf_{t^*_{Z_i} \in AZ_i}$ $sup(g_p(pos(c_i)(t_{X_i},\ t^*_{Z_i})),\ c_S(\pi_{Z_i}(t^*_{Z_i}))) \sqsubseteq inf_{t^*_{Z_i} \in AZ_i}\ sup(g_p(pos(c_i)(t'_{X_i},\ t^*_{Z_i})), c_S(\pi_{Z_i}\ (t^*_{Z_i})))$. Since the map g_p^{-1} is monotone, then $g_p^{-1}(inf_{t^*_{Z_i} \in AZ_i}\ sup(g_p(pos(c_i)$

$(t_{X_i},\ t^*_{Z_i})),\ c_S(\pi_{Z_i}(t^*_{Z_i})))) \ \leq_S\ g_p^{-1}(inf_{t^*_{Z_i} \in AZ_i}\ sup(g_p(pos(c_i)(t'_{X_i},\ t^*_{Z_i})),\ c_S(\pi_{Z_i}$ $(t^*_{Z_i})))))$, i.e., the preferences in the robustness constraints are $defp''_i(t_{X_i}) \leq_S defp''_i(t'_{X_i})$.

By monotonicity of $\times_p$, if we combine via $\times_p$ all such constraints $\langle defp''_i, X\rangle$ we have that, $\prod_{p\ \langle defp''_i, X_i\rangle} defp''_i(t_{X_i}) \leq_S \prod_{p\ \langle defp''_i, X_i\rangle} defp''_i(t'_{X_i})$.

Similarly, it can be shown that $\prod_{\langle defn''_i, X_i\rangle} defn''_i(t_{X_i}) \leq_S \prod_{\langle defn''_i, X_i\rangle} defn''_i(t'_{X_i})$, since the maps g_n and g_n^{-1}, are monotone and since the $\times_n$ operator is monotone.

From here we can conclude as in the proof of Theorem 1.

$\square$

Theorem 7. *If we determine the robustness constraints described in Section 5.1 with the maps g_p, g_p^{-1}, g_n, g_n^{-1}, and c_S such that $\langle g_p, g_p^{-1}\rangle$: (P, $\leq_S$) $\rightleftharpoons$ ($[0,1]$, $\sqsubseteq$) and $\langle g_n, g_n^{-1}\rangle$: (N, $\leq_S$) $\rightleftharpoons$ ($[0,1]$, $\sqsubseteq$) are Galois insertions, and c_S is an order reversing map such that $\forall a, b \in [0,1]$, if $a \leq b$, then $c_S(a) \sqsupseteq c_S(b)$, and $\forall p \in [0,1]$, $c_S(c_S(p)) = p$, the definition of robustness given in Definition 5 satisfies Property BP2.*

Proof. The first part of proof coincides with the one of Theorem 2.

Consider a solution s of the UBCSPs $Q_1 = \langle S, V_c, V_u, \pi_1, C_c, C_{cu}\rangle$ and $Q_2 = \langle S, V_c, V_u, \pi_2, C_c, C_{cu}\rangle$, where $S = \langle N,\ P\ +,\ \times,\ \bot,\ \Box,\ \top\rangle$ is a bipolar preference structure such that P and N are intervals of $\mathbb{R}$ ($\mathbb{Z}$ or $\mathbb{Q}$). Assume that for every assignment t_Z to the uncontrollable variables in V_u, $\pi_2(t_Z) \leq \pi_1(t_Z)$. To prove Property BP2, we will show that $rob_{\pi_1}(s) \leq_S rob_{\pi_2}(s)$, where rob_{π_1} is the robustness computed in the problem with possibility distribution π_1, and rob_{π_2} is the robustness computed in the problem with possibility distribution π_2. The new part starts from here.

Assume the notation considered in the first part of the proof of Theorem 6. By hypothesis, we know that $\forall t_{Z_i} \in AZ_i$, $\pi_2(t_{Z_i}) \leq \pi_1 t_{Z_i}$. By definition of c_S $\forall t_{Z_i} \in AZ_i$, $c_S(\pi_2(t_{Z_i})) \sqsupseteq c_S(\pi_1 t_{Z_i})$. By monotonicity of the sup operator, we have $\forall t_{Z_i} \in AZ_i$, $sup(g_p(pos(c_i)\ (t_{X_i}, t_{Z_i})), c_S(\pi_1(t_{Z_i}))) \sqsubseteq sup(g_p(pos(c_i)(t_{X_i}, t_{Z_i})),\ c_S(\pi_2$ $(t_{Z_i})))$ and $sup(g_p(neg(c_i)\ (t_{X_i}, t_{Z_i})), c_S(\pi_1(t_{Z_i}))) \sqsubseteq sup(g_p(neg(c_i)(t_{X_i}, t_{Z_i})),$ $c_S(\pi_2\ (t_{Z_i})))$. From here we can conclude as in the proof of Theorem 6. $\square$

We now show, via an example, how to instantiate the functions defined above, i.e., g_p, g_p^{-1}, g_n, g_n^{-1}, and c_S, in an UBCSP where the positive and the negative are not defined over intervals. Notice that this UBCSP cannot be solved by the procedure for defining robustness constraints described in Section 5.1, since it is only able to handle UBCSPs where the positive preferences and the negative one are defined over real intervals.

Example 10. Consider an UBCSP $\langle S, V_c, V_u, \pi, C_c, C_{cu}\rangle$, where $S = \langle \mathbb{R}^-, \mathbb{R}^+, max,$ sum, -∞, 0, $+\infty\rangle$. Let us denote with $\leq_S$ the ordering induced by the additive operator of S. To compute robustness constraints we can choose as c_S the map such that $\forall p \in [0,1]$, $c_S(p) = 1 - p$. Moreover, the Galois insertion $\langle g_n, g_n^{-1}\rangle$: ($\mathbb{R}^-$, $\leq_S$) $\rightleftharpoons$ ($[0,1]$, $\leq_\mathbb{R}$), where $\leq_\mathbb{R}$ is the classical order over real numbers, can be defined in different ways. For example, we can use the Galois insertion shown in Example 17 of [5], such that g_n maps all the reals below some fixed real x onto 0 and all the reals over $[x, 0]$ into the reals in $[0,1]$ by using a normalization function $f(r) = (x - r)/x$. Similarly, we can define the Galois insertion $\langle g_p, g_p^{-1}\rangle$: ($\mathbb{R}^+$, $\leq_S$) $\rightleftharpoons$ ($[0,1]$, $\leq_\mathbb{R}$), assuming that g_p maps all the reals above some fixed real x onto 1 and all the reals over $[0, x]$ into the reals in $[0,1]$ by using the same normalization function considered before, i.e., $f(r) = (x-r)/x$. $\square$

9 Conclusions and future work

We have considered problems with bipolar preferences and uncontrollable variables, and with a possibility distribution over such variables (UBCSPs). We have then defined the notion of preference and robustness for such problems, as well as some desirable properties that such notions should respect, also in relation to the solution ordering. By following the approach shown in [26] for problems with fuzzy preferences and uncertainty, we have provided an algorithm for UBCSPs, that removes the uncontrollable part of the problem while altering the controllable part in order to loose little information. On the resulting problem, we have then defined the preference and the robustness of a solution of the initial UBCSP. Different semantics use such two notions to order the solutions according to different attitudes to risk. We have then shown that our proposed notions of preference and robustness, as well as our semantics, satisfy the desired properties we have considered.

We have first considered UBCSPs where the sets of positive and negative preferences are closed real intervals, and then we have generalized the proposed approach to the case of generic totally ordered preferences by using abstraction techniques and Galois connections.

The results of the paper show that it is possible, without much effort, to deal simultaneously with possibilistic uncertainty and bipolar preferences, while making sure that several desirable properties hold and without requiring a bipolarization of the possibility scale. In other words, our results state that it is possible to extend the formalism in [8, 10] to bipolar preferences and the one in [25] to uncertainty, while preserving the desired properties.

Following this approach, a solver for UCSPs would thus first remove the uncontrollable part, and then find an optimal solution of the controllable part according to a chosen semantics. Such a solver may be developed by adapting constraint propagation and branch and bound techniques that have been already defined and implemented for bipolar CSPs in [8, 10].

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