Performance analysis of magnetic recording systems

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Abstract—**Approximations to the union bound performance of sequence detection in the presence of colored noise and an algorithm to compute bit error and error event probabilities are presented and compared to bit-by-bit simulation results. These computations, which are very accurate at bit error probabilities $\leq 10^{-3}$, are then used to analyze the performance of standard and reverse concatenated Reed-Solomon/modulation coding schemes for generalized partial response channels corrupted by colored noise. The analysis is used to determine the optimum RS code rate for recording systems that are of current interest.**

Keywords—**Magnetic recording, performance analysis, colored noise.**

I. Introduction

PARTIAL response class 4 (PR4) and extended PR4 channels (see [5] and references therein) have been the state of the art in disk drives until quite recently. At high linear densities generalized partial response polynomials with real coefficients provide a better match to the discrete-time response of the recording channel than monic polynomials with integer coefficients. This class of polynomials, when combined with sequence detection, gives rise to noise-predictive maximum likelihood (NPML) systems [8], [6]. Currently, 16-state NPML detectors for generalized partial response channels with a first order null at DC operating at rates close to 1 Gbit/s represent the state of the art in the disk drive industry.

Error control coding has played an important role in the design of the overall recording system. In disk drives the usual coding scheme for partial response recording channels is concatenated coding with an outer Reed-Solomon (RS) code and an inner modulation/parity code [4]. In this type of coding scheme, which is also known as standard concatenation,the use of high-rate modulation codes with block sizes varying from say two to eight bytes leads to either weak code constraints or an increase of error propagation at the modulation decoder. This fact coupled with the desire to perform soft-decision decoding has led to renewed interest in reverse concatenation [2]. For a PR4 sequence detector and a concatenated coding scheme based on RS codes with 8-bit symbols and rate-16/17 modulation codes, reverse concatenation permits the use of three interleaved RS codewords per sector, whereas standard concatenation requires at least four interleaved RS codewords per sector to achieve good overall performance [9].

Union bounds for intersymbol interference channels that estimate the performance of a sequence detector in the presence of white and colored Gaussian noise have been derived in [10] and [11], respectively. An exact computation of these union bounds using generating-function methods similar to those employed in [10] and [11] is possible only in the case of white Gaussian noise. This is due to the fact that the noise variance associated with an error event in the presence of colored noise, also known as the modified variance, depends on the particular error event under consideration. In this paper, we present approximations to the union bound in [11] and an algorithm to compute bit error and error event probabilities for sequence detection in the presence of colored noise. These probabilities are then used to evaluate the overall performance of a recording system by following an analytic approach similar to the one given in [9].

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The paper is organized as follows. In Section II, the recording system model including error correction coding and modulation coding is described and the equivalent discrete-time model of the recording channel is introduced in form of a generalized partial response channel impaired by colored noise. In Section III, methods of computing approximations to the bit error and error event probabilities at the output of the sequence detector are developed and compared to results obtained by computer simulation. In Section IV, the performance of standard and reverse concatenated RS/modulation coding schemes for generalized partial response channels corrupted by colored noise is analyzed. In particular, the optimum RS code rate is determined for the coding schemes that are usually used in disk drives.

II. Recording System Model

A block diagram of the recording system model considered in this paper is shown in Fig. 1. The outer RS encoder operates on 8-bit symbols (bytes) and can correct up to t bytes per RS codeword. A 512-byte sector is encoded into $I = 3$ or 4 RS codewords that are byte-interleaved and fed to the modulation encoder. The inner code is a high-rate modulation code that imposes constraints to aid timing/gain recovery and avoid quasi-catastrophic error propagation at the output of the sequence detector. Clearly, the desirable properties of the inner modulation code is high code rate and small error bursts at the modulation decoder output.

The binary outputs of the modulation encoder are mapped into bipolar symbols +1 and -1, which are written onto the disk in form of a positive or negative magnetization along a circular track. The data sequence is read back from the head/disk assembly as an analog signal. After low-pass filtering and sampling, the signal is shaped into a partial response signal format by the equalizer. The power of the total distortion at the output of the equalizer is reduced by noise prediction [3], [8]. Adopting a linear model for the read/write process, the chain of signal processing functions including read/write heads, preamplifier, automatic gain control, low-pass filtering, sampling, equalization and noise whitening is modeled by the generalized partial response polynomial $F(D) = 1 + \sum_{i=1}^{L} f_i D^i$ where L is the channel memory and f_i are real coefficients.

Figure 2 depicts the equivalent discrete-time model of the recording channel extending from the output of the modulation encoder to the input of the sequence detector. The input data sequence $\{x_k\}$ to the generalized partial response channel is bipolar, i.e., $x_k \in \{-1,+1\}$. The sequence of colored noise samples $\{\eta_k\}$ at the input of the sequence detector are generated by passing the sequence of white noise samples $\{n_k\}$ through an equivalent filter $h(D)$ that represents the combination of the low-pass filter, the equalizer and the whitening filter. Finally, the sequence detector provides delayed estimates $\{\hat{x}_k\}$ of the bipolar symbols that have been fed to the recording channel.

III. DETECTOR PERFORMANCE IN THE PRESENCE OF COLORED NOISE

Let $\mathcal{E}$ be an error event of length i that is characterized by the input error sequence $\epsilon_x(D) = x(D) - \hat{x}(D)$ of length i. As the generalized partial response channel is linear and time-invariant, the output error sequence can be expressed as $\epsilon_y(D) = \epsilon_x(D)F(D)$. The squared Euclidean distance associated with a particular error event $\mathcal{E}$ of length i is defined as the energy of the corresponding output error sequence of length $i+L$, i.e.,

$$d^2(\mathcal{E}) = ||\epsilon_y||^2. \tag{1}$$

At the output of a sequence detector let E_i denote the set of all error events of length i and b_i denote the probability that an error burst of length i starts at a particular time instant. The union bound states that the probability of a union of events is less than or equal to the sum of their individual probabilities. Consequently,

$$b_i = \Pr(E_i) \leq \sum_{\mathcal{E}\in E_i} \Pr(\mathcal{E}). \tag{2}$$

Similarly, an upper bound on the probability of bit error P_b at the output of the sequence detector is given by

$$P_b \leq \sum_{\mathcal{E}\in E} w(\mathcal{E})\Pr(\mathcal{E}) \tag{3}$$

where $w(\mathcal{E})$ is the Hamming weight of the error event $\mathcal{E}$, which is defined as the number of nonzero coefficients in the corresponding input error sequence, $\Pr(\mathcal{E})$ is the probability of the error event $\mathcal{E}$ and E is the set of all possible error events.

In the following we classify the error events by their Euclidean distance. Let $\mathcal{E}^{(u,m)}$ be the m-th error event with distance d_u. Furthermore, $w_{u,m}$ denotes the Hamming weight of the error event $\mathcal{E}^{(u,m)}$ and $n(u,m)$ denotes the length of the corresponding output error sequence. Now we define the vector $V_{u,m}$ as the column vector corresponding to the normalized output error sequence, i.e.,

$$V_{u,m} = \frac{\epsilon_y(\mathcal{E}^{(u,m)})}{||\epsilon_y(\mathcal{E}^{(u,m)})||}. \tag{4}$$

We now reformulate the bit error and error event probability results derived in [11] by using the error event classification that we have introduced. For this purpose the modified variance $\sigma_{u,m}^2$ associated with an error event $\mathcal{E}^{(u,m)}$ is defined as the variance along the vector $V_{u,m}$. The modified variance is given by the quadratic form

$$\sigma_{u,m}^2 = V_{u,m}^T \Phi V_{u,m}, \tag{5}$$

where Φ is the $n(u,m)\times n(u,m)$ covariance matrix associated with the sequence of colored noise samples $\{\eta_k\}$.

The bit error probability of the sequence detector is then bounded by

$$P_b \leq \sum_u \sum_m w_{u,m} Q\left(\frac{d_u}{2\sigma_{u,m}}\right) 2^{-w_{u,m}}, \tag{6}$$

where $Q(\cdot)$ represents the tail integral of the zero-mean, unit variance normal distribution and $w_{u,m}$ denotes the Hamming weight of the m-th error event with distance d_u. Similarly, the error burst distribution b_i can be bounded by

$$b_i \leq \sum_u \sum_m Q\left(\frac{d_u}{2\sigma_{u,m}}\right) 2^{-w_{u,m}}, \tag{7}$$

where now only error events $\mathcal{E}^{(u,m)} \in E_i$ that have distance d_u and length i are considered.

A. Approximations to the union bound

For maximum-likelihood sequence detection in the presence of white Gaussian noise it is common practice to obtain first order approximations to union bounds by considering only error events with minimum distance. Clearly, tighter bounds can be obtained if the computations also include error events that have distance larger than the minimum distance. In the case of colored noise, the upper bounds on bit error and error event probabilities in Eqs. (6) and (7) are also obtained as a sum over infinitely many terms. However, the contribution of each term in Eqs. (6) and (7) is mainly determined by the ratio $\frac{d_u}{2\sigma_{u,m}}$ associated with an error event rather than just the Euclidean distance of an error event as in the case of white noise. Consequently, the following approximations to the union bound consider only error events that satisfy the criterion

$$\frac{d_u}{2\sigma_{u,m}} \leq q_{max}, \tag{8}$$

where q_{max} is a predetermined quantity that should be selected judicously. We remark that, while the selection of error events according to their distance can be done efficiently in the white noise case by exploring error flow graphs associated with the channel polynomial [10], [11], there is no simple solution for evaluating the modified variance. Therefore, to further simplify the selection of error events an upper bound on the modified variance that depends on the length of the output error sequence is obtained. It can be shown that the following upper bound on the modified variance results from applying Schwarz's inequality to the quadratic form in (5)

$$\sigma_{u,m}^2 \leq \sigma^2 n(u,m)||h||^2, \tag{9}$$

where σ^2 is the variance of the white noise $\{n_i\}$. The first sum in Eq. (6) can now be restricted only to error events that satisfy the condition

$$d_u \leq d_{max} = 2q_{max}\sigma\sqrt{n(u,m)}||h||, \tag{10}$$

and the resulting probability of bit error is

$$P_b \leq \sum_{u|d_u\leq d_{max}} \sum_m w_{u,m} Q\left(\frac{d_u}{2\sigma_{u,m}}\right) 2^{-w_{u,m}}. \tag{11}$$

Similarly, the burst error distribution b_i can be approximated as

$$b_i \leq \sum_{u|d_u \leq d_{max}} \sum_m Q\left(\frac{d_u}{2\sigma_{u,m}}\right) 2^{-w_{u,m}}, \quad (12)$$

where again only error events $\mathcal{E}^{(u,m)}$ with distance d_u and length i are considered.

Summation over m in the above equations may involve an infinite number of terms with the same distance due to the presence of one or more zero-cycles [1]. A zero-cycle is a subevent $\epsilon_y(D)$ that can be repeated any number of times without accumulating Euclidean distance. There may be many zero-cycles in an error event. In general, the zero-cycles occur if the frequency response of the channel has spectral nulls. Magnetic recording channels have a spectral null at dc and therefore exhibit zero-cycles.

In the following we will call error events that do not contain zero-cycles elementary error events. Let us assume that n_{cyc} zero-cycles are added to an elementary error event, i.e. the input error sequence is

$$\epsilon_x = \left[\epsilon_{x,1}\, \hat{\epsilon}_{x,1}\, \ldots\, \epsilon_{x,n_{cyc}}\, \hat{\epsilon}_{x,n_{cyc}}\, \epsilon_{x,n_{cyc}+1}\right],$$

where $\epsilon_{x,m}$ is an elementary subevent and $\hat{\epsilon}_{x,m}$ is a subevent made of the repetition of a zero-cycle.

As the filter $h(D)$ has finite length N_{eq}, $E[\eta_k \eta_i] = 0$, for $|k-i| > N_{eq}$, therefore $[\Phi]_{r,c} = 0$ for $|r-c| > N_{eq}$. Now, if the output error sequence is zero between the time instants k_1 and k_2 due to the presence of zero cycles, and $k_2 - k_1 > N_{eq}$, the modified variance of the noise can be expressed as

$$\begin{aligned} \sigma^2_{u,m} &= \sum_{r=1}^{k_1}\sum_{c=1}^{k_1}[V_{u,m}]_r[V_{u,m}]_c[\Phi]_{r,c} \quad (13)\\ &+ \sum_{r=k_2}^{n(u,m)}\sum_{c=k_2}^{n(u,m)}[V_{u,m}]_r[V_{u,m}]_c[\Phi]_{r,c}, \end{aligned}$$

and this expression is independent of $|k_2 - k_1|$, the length of the zero-cycle. Therefore, if there is a sufficient number of zero-cycles, the argument of $Q(\cdot)$ in the above equations does not change.

Now let $\hat{w}_{u,m}$ denote the Hamming weight of a zero-cycle and $\sigma_{u,m}$ denote the modified variance of an elementary error event with Hamming weight $w_{u,m}$. Furthermore, we define $\hat{\chi}_{u,m} = 2^{-\hat{w}_{u,m}}$ and $\chi_{u,m} = 2^{-w_{u,m}}$. Finally, let $\mathcal{S}$ be the set of all two-tuples (u,m) such that (u,m) is associated with an elementary error event with Euclidean distance $d_u \leq d_{max}$. Then, by induction, the summation in (11) can be rewritten as follows

$$\begin{aligned} P_b &\leq \sum_{(u,m)\in\mathcal{S}} Q\left(\frac{d_u}{2\sigma_{u,m}}\right)\left(\frac{w_{u,m}\chi_{u,m}}{1-\hat{\chi}_{u,m}}\right.\\ &\left. + \frac{\hat{w}_{u,m}\chi_{u,m}\hat{\chi}_{u,m}}{(1-\hat{\chi}_{u,m})^2}\right). \quad (14) \end{aligned}$$

Similarly, the burst error distribution can be expressed as

$$b_i \leq \sum_{(u,m)\in\mathcal{S}} Q\left(\frac{d_u}{2\sigma_{u,m}}\right)\left(\frac{\chi_{u,m}}{1-\hat{\chi}_{u,m}}\right), \quad (15)$$

where the summation is over elementary error events with distance $d_u \leq d_{max}$ and length i.

B. A search algorithm for evaluating detector performance

In this subsection we give an algorithm that performs a modified depth-first search in an error flow graph. The search procedure identifies the error events that will be used to compute bit error and error event probabilities. Search algorithms for characterizing error events with specified distance at the output of partial-response channels of the form $(1-D)^m(1-D)^n,\ m,n \geq 0$ have been presented in [1].

The error flow graph is a directed graph that may be constructed in the following manner. The states are defined as the difference between the actual state of the generalized partial response channel and the state estimated by the sequence detector. The edges connect nodes such that the first $L-1$ elements of the initial state coincides with the last $L-1$ elements of the terminal state. Each edge going from state $[\epsilon_x(k-1), \epsilon_x(k-2), \ldots, \epsilon_x(k-L)]$ to state $[\epsilon_x(k), \epsilon_x(k-1), \ldots, \epsilon_x(k-L+1)]$ is weighted with distance

$$(\epsilon_x(k) + \sum_{i=1}^{L}\epsilon_x(k-i)f_i)^2.$$

It is well known that the error flow graph is symmetric, i.e., the distance associated with an edge going from state $[\epsilon_x(k-1), \epsilon_x(k-2), \ldots, \epsilon_x(k-L)]$ to state $[\epsilon_x(k), \epsilon_x(k-1), \ldots, \epsilon_x(k-L+1)]$ is the same as that of the edge going from state $[-\epsilon_x(k-1), -\epsilon_x(k-2), \ldots, -\epsilon_x(k-L)]$ to state $[-\epsilon_x(k), -\epsilon_x(k-1), \ldots, -\epsilon_x(k-L+1)]$. This property of error flow graphs can be used to simplify the computations.

The algorithm starts from the zero state [0, 0, …, 0] and extends gradually the paths considering the successors of the zero state and then the successors of the successors and so on. Let us refer to the current path as the temporary path. The condition $\frac{d_u+d_{u,end}}{2\sigma_{u,m}} \leq q_{max}$ is checked on each path extension, where $d_{u,end}$ is the minimum distance of the subpath going from the ending state of the temporary path to the zero state and $\sigma^2_{u,m}$ is the modified variance associated with the temporary path. The computation of $d_{u,end}$ is performed at the beginning of the algorithm, using a modified version of Dijkstra's algorithm [7]. Not all the temporary paths are considered and in particular the process of exploration of the successors stops when one of the following conditions is satisfied.

1. If the zero state is reached, the path is used for the computation of P_b from Eq. (14) and b_i from Eq. (15).
2. If the condition on the distance-to-modified variance ratio is not satisfied, the temporary path is discarded.
3. If the length of a zero cycle repetition at the end of the temporary path exceeds N_{eq}, $\hat{w}$ and $\hat{\chi}$ are computed and no more cycles are allowed to grow at the end of the temporary path.

C. Simulation results

The search algorithm presented in the previous section will be used to compute the bit error probabability at the output of a Viterbi detector for a degree-4 generalized partial response channel that is corrupted by colored noise. The channel is characterized by the polynomial $F(D) = (1-D^2)(1+p_1D+p_2D^2)$

where it has been assumed that the target of the 10-coefficient equalizer is $(1 - D^2)$ and the predictor has two coefficients that have been optimized to minimize the noise power at the input of the detector. The read/write process extending from the output of the modulation encoder to the input of the low-pass filter has been modeled as a Lorentzian channel corrupted by additive white Gaussian noise. Consequently, the noise at 16-state the detector input can be represented as filtered white noise where the filter corresponds to the combination of the low-pass filter, the 10-coefficient equalizer and the whitening filter $(1 + p_1 D + p_2 D^2)$ with two predictor coefficients.

Figure 3 shows the bit error probability as a function of the signal-to-noise ratio (SNR) for a Lorentzian channel with normalized linear density $PW50/T = 2.5$. The bit-by-bit simulation results agree well with the computed bit error probabilities. The results demonstrate that the computed approximations to the union bound are very accurate for $P_b \leq 10^{-3}$. It is well known that union bounds are not tight at bit error probabilities $P_b > 10^{-3}$. Similarly, Fig. 4 shows the bit error probability as a function of the normalized linear density $PW50/T$ for SNR=14dB. Again we observe good agreement between the analytical results and bit-by-bit simulations.

IV. Recording system performance

The performance of concatenated RS/modulation coding has been analyzed in [9] as a function of the code parameters and the polynomial

$$\tilde{b}(z) = \frac{1}{\sum_{i=1} b_i} \sum_{i=1} b_i z^i, \qquad (16)$$

that characterizes the burst error distribution at the input of the modulation decoder. This analysis applies to both standard and reverse concatenation and employs the sum of the probability of decoder error and decoder failure $P_{decoder}$ as the performance measure. As an RS decoder error can usually be detected with additional cyclic redundancy check coding, $P_{decoder}$ is equivalent to the reread probability in disk drives, the probability that the sector is read for a second time after a delay of one revolution.

In this section the performance of concatenated RS/modulation coding (standard and reverse) for generalized partial response channels will be analyzed based on the algorithm for computing the polynomial $\tilde{b}(z)$ that has been discussed in the previous section. Specifically, we will compute the probability

$$P_{decoder} = \sum_{i=1}^{N_{sector}} \binom{N_{sector}}{i} b^i (1-b)^{N_{sector}-i} \left(\sum_{j=t+1}^{\infty} (q(z)^i)_j \right), \qquad (17)$$

where $b = \sum_{i=1} b_i$, N_{sector} is the total number of recorded bits per sector and $q(z)$ is a polynomial that can be expressed as a function of $\tilde{b}(z)$ [9]. The subscript notation $(q(z)^i)_j$ indicates that the $j-$th coefficient of the polynomial $q(z)^i$ is used.

In the following we assume that the degree-4 generalized partial response channel described in Section IIIC is used. Furthermore, a rate 16/17 modulation code is employed in standard concatenation where it is assumed that errors at the input of the modulation decoder cause two bytes to be in error. For reverse concatenation the same modulation code is used to encode user bytes whereas a rate 8/9 modulation code is used to encode RS parity bytes. At a normalized user density $PW50/T_u = 2.6$ Fig. 5 compares the performance of standard and reverse concatenation schemes for $I = 3$ and 4 interleaves per sector where each interleave can correct $t = 6$ bytes. After accounting for the rate increase due to RS and modulation coding the normalized channel density $PW50/T$ is about 3. It can be seen that reverse concatenation permits the use of three interleaved RS codewords per sector, whereas standard concatenation requires at least four interleaved RS codewords per sector to achieve good overall performance. Finally, Fig. 6 shows $P_{decoder}$ as a function of t for standard and reverse concatenation schemes with $I = 3$ and 4 interleaves per sector where again $PW50/T_u = 2.6$. The SNR values have been selected such that the minimum of $P_{decoder}$ is in the range of 10^{-11} to 10^{-10}. It can be seen that in all four cases the optimum RS code rate is about 0.86.

V. Conclusions

The bit error and error event probabilities of Viterbi detectors for generalized partial response channels that are corrupted by colored noise have been computed using approximations to union bounds and a modified depth-first search algorithm. The performance of standard and reverse concatenated RS/modulation coding schemes for generalized partial response channels has been analyzed using the sum of RS decoder error and RS decoder failure probabilities as the performance measure. Computations have shown that the optimum RS code rate for current disk drive designs is about 0.86.

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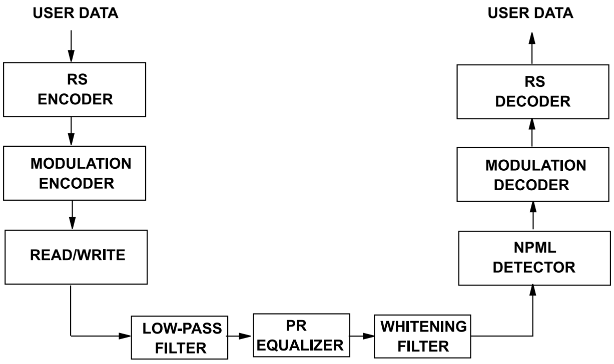

Fig. 1. Recording system model

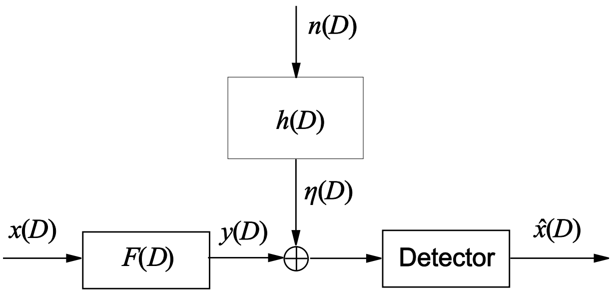

Fig. 2. Equivalent discrete-time model for the inner channel

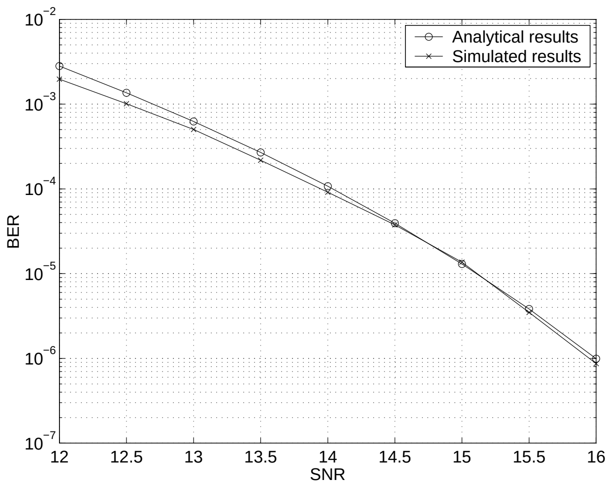

Fig. 3. Bit error probability as a function of SNR for $PW50/T = 2.5$.

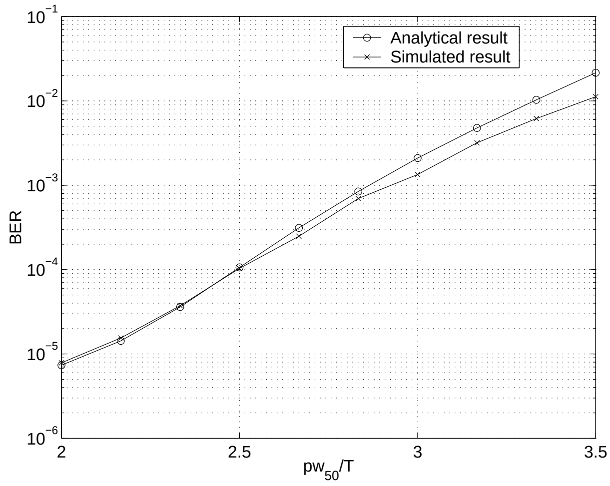

Fig. 4. Bit error probability as a function of $PW50/T$ for SNR$= 14$ dB.

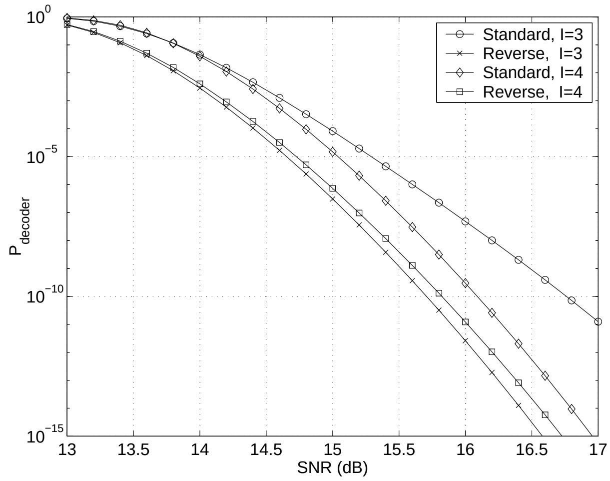

Fig. 5. Performance comparison for $t = 6$, $PW50/Tu = 2.6$.

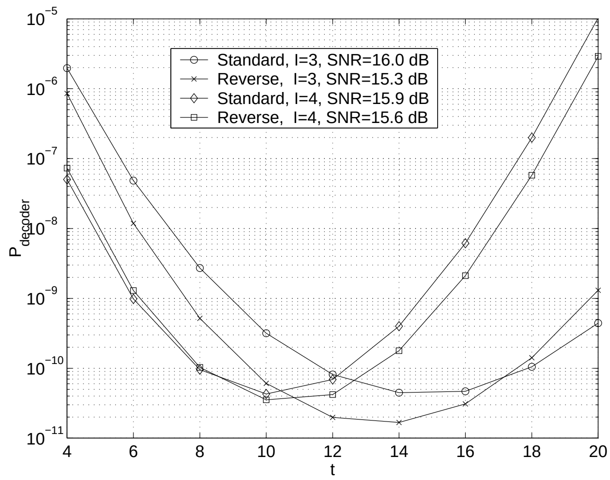

Fig. 6. Optimum RS code rate for $PW50/Tu = 2.6$.